POYNTING THEOREM →

To find the power in a uniform plane wave in 1884 John H. poynting developed a power theorem for Electromagnetic field using maxwell's equations known as poynting Theorem.

Statement →

In a uniform plane wave at any point, product of electric field intensity $\vec{E}$ & magnetic field intensity $\vec{H}$ is a measure of rate of energy flow at the point.

In Mathematical form

$$\vec{P} = \vec{E} \times \vec{H}, \qquad \vec{P} = \text{Poynting vector}$$

or

$$\oint_S (\vec{E} \times \vec{H})\, d\vec{s} = -\int_V (\vec{E} \cdot \vec{J})\, dV - \frac{\partial}{\partial t} \int_V \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) dV$$

Rate of energy flow powerloss rate of decrease of stored energy.

From Maxwell's 3^{rd} and 4^{th} equation →

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t} \quad \text{— (1)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \quad \text{— (2)}$$

Now taking dot product with $\vec{E}$ in (1) equation:

$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}$ —(3)

using vector identity:

$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$

$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{H})$

Put value in equation (3)

$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}$

$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}$ —(4)

Now take dot product with $\vec{H}$ in equation (2)

$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) = -\left(\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}\right)\mu.$

Put this value in eqn (4)

$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}\right)\mu - \vec{E}\cdot\vec{J} - \left(\vec{E}\cdot\frac{\partial \vec{E}}{\partial t}\right)\epsilon$ —(5)

$(\because \vec{D} = \epsilon\vec{E})$

Let us consider the term

$\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = \vec{H}\cdot\frac{\partial \vec{H}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{H}}{\partial t}.$

$\frac{\partial}{\partial t}(H^2) = 2\vec{H}\cdot\frac{\partial \vec{H}}{\partial t}.$

$\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}H^2$

Similarly, $$\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}$$

Put these values in equation (2).

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \vec{E}\cdot\vec{J} - \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t}$$

Taking volume integral on both sides

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = -\frac{\mu}{2}\int_V \frac{\partial H^2}{\partial t}\,dV - \int_V (\vec{E}\cdot\vec{J})\,dV - \frac{\varepsilon}{2}\int_V \frac{\partial E^2}{\partial t}\,dV.$$

Applying Gauss divergence theorem

$$\oint_S (\vec{E}\times\vec{H})\,ds = -\frac{\partial}{\partial t}\int_V \frac{\mu H^2}{2}\,dV - \int_V (\vec{E}\cdot\vec{J})\,dV - \frac{\partial}{\partial t}\int_V \frac{\varepsilon E^2}{2}$$

$$\boxed{\oint_S (\vec{E}\times\vec{H})\,ds = -\int_V (\vec{E}\cdot\vec{J})\,dV - \frac{\partial}{\partial t}\int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dV}$$