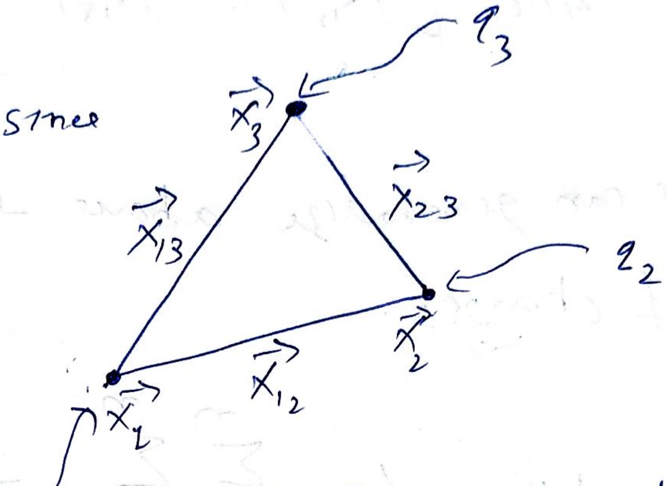
(I). Energy of a Point charge distribution:

For charge $q_1$ work done $= 0$, since there is no field present.




Bringing charge $q_2$, work done will be $q_2 V_1(\vec{x}_2)$,

$V_1 \rightarrow$ Potential due to $q_1$.

In Fig. we bring charges one by one from far away

$\vec{x}_2 \rightarrow$ denotes the position where we put charge $q_2$.

Now $\quad W_1 = 0 \quad$ ——— (1)

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left( \frac{q_1}{|\vec{x}_{12}|} \right) \quad \text{——— (2)}$$

Similarly, for bringing charge $q_3$, work done

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left( \frac{q_1}{|\vec{x}_{13}|} \right) + \frac{1}{4\pi\epsilon_0} q_3 \frac{q_2}{|\vec{x}_{23}|}$$

or
$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left[ \frac{q_1}{|\vec{x}_{13}|} + \frac{q_2}{|\vec{x}_{23}|} \right] \quad \text{—— (3)}$$

Similarly for any charge $q_4$, work done

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left[ \frac{q_1}{|\vec{x}_{14}|} + \frac{q_2}{|\vec{x}_{24}|} + \frac{q_3}{|\vec{x}_{34}|} \right] \text{—— (4)}$$

Now the total work done to assemble the first four charges will be given by

$$W = W_1 + W_2 + W_3 + W_4$$

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\vec{x}_{12}|} + \frac{q_1 q_3}{|\vec{x}_{13}|} + \frac{q_1 q_4}{|\vec{x}_{14}|} + \frac{q_2 q_3}{|\vec{x}_{23}|} + \frac{q_2 q_4}{|\vec{x}_{24}|} + \frac{q_3 q_4}{|\vec{x}_{34}|}\right] \quad \text{--- (5)}$$

We can generalize above expression for n number of charges.

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_i q_j}{|\vec{x}_{ij}|}$$

{ $j>i$ make sure that no pair is repeated

or we can write above expression

$$W = \frac{1}{2}\left[\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{|\vec{x}_{ij}|}\right]$$

{ $\frac{1}{2}$ factor is multiplied so that repeated pair ~~does not~~ does not affect the final result

$$\boxed{W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{q_i q_j}{|\vec{x}_{ij}|}} \quad \text{--- (6)}$$

$$\Downarrow$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{|\vec{x}_{ij}|}\right]} \quad \text{--- (7)}$$

Term in the square bracket represents the potential at point $\vec{x}_i$, due to all other charges $q_j$. Therefore, we can write

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{x}_i)} \quad \text{--- (8)}$$

($E_2$) Energy of a continuous charge distribution:

From (8), we see that for volume charge density $\rho$, this can be written as.

$$W = \frac{1}{2}\int \rho V \, d\tau \quad \text{—— (9)}$$

for surface charge and line charges, $W$ is given by respectively. $\int \sigma V \, da$ and $\int \lambda V \, dl$.

The Gauss's law state that $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\Rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E} \quad \text{—— (10)}$$

Eq. (9) can be written as

$$W = \frac{1}{2}\int \epsilon_0 \nabla \cdot \vec{E} \, V d\tau = \frac{\epsilon_0}{2}\int (\nabla \cdot \vec{E}) V d\tau$$

Now using integration by parts in the above equation

$$W = \frac{\epsilon_0}{2}\left[-\int \vec{E} \cdot \nabla V d\tau + \oint V \vec{E} \cdot d\vec{a}\right]$$

Since $\nabla V = -\vec{E}$, therefore we can write

$$W = \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V \vec{E} \cdot d\vec{a}\right]$$

Integrating over all space, the second term, surface integral, becomes zero. and we have

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau} \quad \text{—— (11)}$$