Green's Theorem

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"One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we originally put in to them."

— Heinrich Hertz (1857-1894)

In the earlier lecture notes, we have discussed divergence theorem. Here we discuss Green's theorem which is an useful corollary of the divergence theorem.

Let u and v be two scalar functions of space having continuous derivative within a certain region bounded by a closed surface a. We can write two identities as

$$\nabla \cdot (u\nabla v) = u(\nabla \cdot \nabla v) + \nabla u \cdot \nabla v \ , \tag{1}$$

$$\nabla \cdot (v\nabla u) = v(\nabla \cdot \nabla u) + \nabla v \cdot \nabla u \ . \tag{2}$$

Next, taking Eq. (1) and integrating over volume, we can write

$$\iiint_{\mathcal{V}} \nabla \cdot (u\nabla v)\, d\tau = \iiint_{\mathcal{V}} \Big(u(\nabla \cdot \nabla v) + \nabla u \cdot \nabla v \Big)\, d\tau \ . \tag{3}$$

Applying divergence theorem on the rhs of above equation we obtain

$$\boxed{\oint\!\!\!\oint_{\mathcal{A}} (u\nabla v) \cdot d\mathbf{a} = \iiint_{\mathcal{V}} u(\nabla \cdot \nabla v)\, d\tau + \iiint_{\mathcal{V}} \nabla u \cdot \nabla v\, d\tau} \ . \tag{4}$$

The above form of the equation is known as the *Green's theorem in the first form*

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The another form of Green's theorem (second form) is obtained subtracting Eq. (1) from Eq. (2) and integrating over the volume. Now subtracting Eq. (1) from Eq. (2), we obtain

$$\nabla \cdot (u\nabla v - v\nabla u) = u(\nabla \cdot \nabla v) - v(\nabla \cdot \nabla u) \ . \tag{5}$$

Next, integrating over volume, we obtain

$$\iiint_{\mathcal{V}} \nabla \cdot (u\nabla v - v\nabla u) = \iiint_{\mathcal{V}} u(\nabla \cdot \nabla v)\, d\tau - \iiint_{\mathcal{V}} v(\nabla \cdot \nabla u)\, d\tau \ . \tag{6}$$

using divergence theorem on the left-hand side of the above equation, we obtain

$$\boxed{\oint\!\!\!\!\oint_{\mathcal{A}} (u\nabla v - v\nabla u) \cdot d\mathbf{a} = \iiint_{\mathcal{V}} u(\nabla \cdot \nabla v)\, d\tau - \iiint_{\mathcal{V}} v(\nabla \cdot \nabla u)\, d\tau} \ . \tag{7}$$

The above equation is the second form of Green's theorem.

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