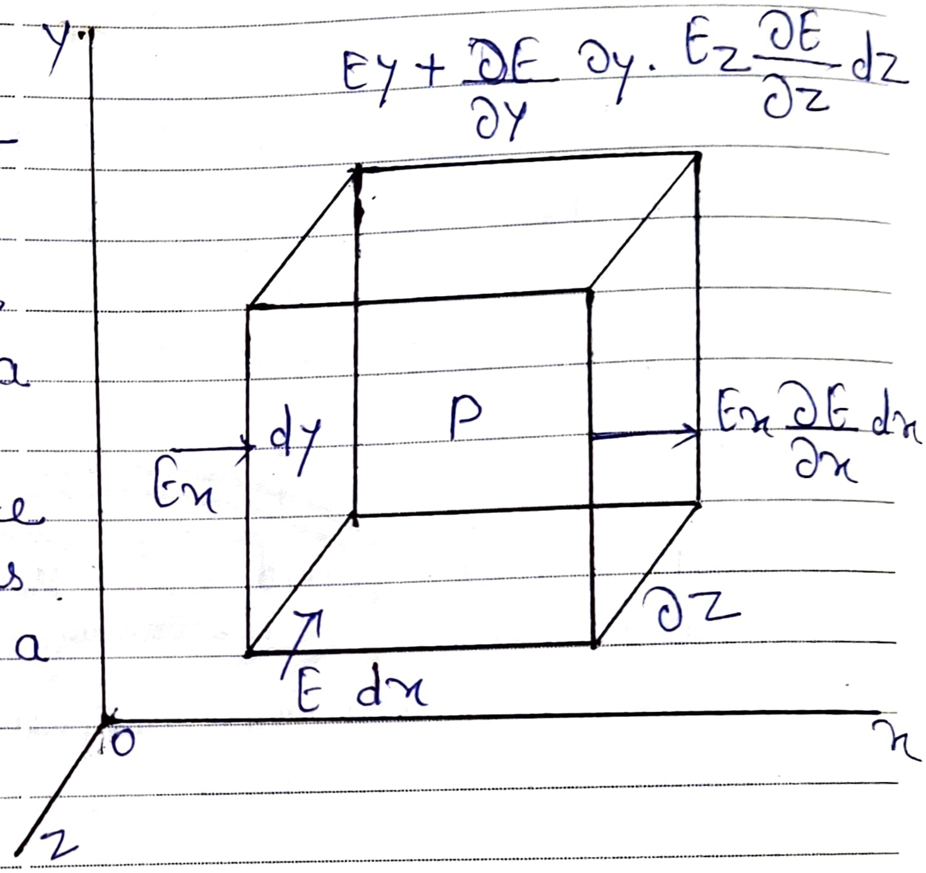
Ques:- Poission's and Laplace's equation for an electric field. (In carlesion co-ordinate) :—



Ans:- Let us suppose that ox, oy and oz are co-ordinate axis and P is a point whose co-ordinate are (x, y, z) Let us suppose that a rectangular parallelopiped of side dx, dy and dz has been drawn surrounding the point P.

Let us suppose that K be the dielectric constant of the medium In which it immersed and ρ. the volume density of charge near P.

Let D be the electric displacement at P. The components of D in the direction of ox, oy and oz are D_x, D_y and D_z respectively and the corresponding electric intensity is D/K whose components in the above direction are E_x, E_y and E_z respectively. Thus $E_x = D_x K$, $E_y = D_y/K$ and

$$E_z = D_z/K$$

The front face is parallel to x, y plane and its area is dx, dy.

The flux of total normal electric displacement. On it will be due to the z component of D. If the point P lies at the centre of the parallelopiped then the co-ordinate of the centre of the above face will be $\left(x, y, z + \frac{dz}{2}\right)$

The rate of change of D_z with respect to z will be

$\frac{\partial D_z}{\partial z}$. Hence the magnitude of D_z at the centre of the face will be $\left(D_z + \frac{\partial D_z}{\partial z} \cdot \frac{\partial z}{z}\right)$

Therefore the flux of normal electric displacement on this face will be

$$\left(D_z + \frac{\partial D_x}{\partial z} \cdot \frac{\partial z}{z}\right) dx, dy$$

Similarly the flux of normal electric displacement on the back parallel to it will be $\left(D_z - \frac{\partial D_z}{\partial z} \cdot \frac{\partial z}{z}\right) dx dy$

Hence the net normal electric displacement in the direction parallel to ox

$$\left(D_z + \frac{\partial D_z}{\partial z} \cdot \frac{\partial z}{z}\right) dx dy - \left(D_z - \frac{\partial D_z}{\partial z} \cdot \frac{\partial z}{z}\right)$$

$$dx\, dy = \frac{\partial D_z}{\partial z}\, dx\, dx\, dy$$

Similarly the net flux of the total normal electric displacement on the faces perpendicular to oy and oz axes will be $\frac{\partial D_z}{\partial x} dx$

$dy\,dz$ and $\frac{\partial D_y}{\partial y} dy\,dx\,dz$ respectively

Hence the flux of total normal electric displacement on the whole surface of the parallelopiped will be

$$= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx, dy, dz$$

$\because$ The volume of the parallelopiped $= dx\,dy\,dz$

$\therefore$ The total charge inside the parallelopiped $= \rho\,dx\,dy\,dz$.

But according to Gauss's theorem, the total normal electric induction over the whole closed surface of the ractangular parallelopiped is equal to $1/\epsilon_0$ times the total charge enclosed by the surface

$$\therefore \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\, dy\, dz =$$

$$= \rho\, dx \cdot dy \cdot dz / \epsilon_0$$

or, $$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho/\epsilon_0 \quad \text{——} \; (i)$$

But $D_x = KE_x$, $D_y = KE_y$ and $D_z = KE_z$

$\therefore$ eqn (1) reduces to

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho / \epsilon_0 \cdot k \quad \text{——} \; (II)$$

In the vector form equation (1) and (II) can be written as

$$\text{div } \vec{D} = \nabla \cdot \vec{D} = \rho / \varepsilon_0 \quad \text{——— (III)}$$

and $\text{div } \vec{E} = \nabla \cdot \vec{E} = \rho / \varepsilon_0 k$ ——— (IV)

For air $k = 1$

∴ Equation (2) becomes

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho / \varepsilon_0 \quad \text{——— (V)}$$

and equation (IV) becomes

$$\text{div } \vec{E} = \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \text{——— (VI)}$$

If $v(x, y, z)$ be the potential at p then

$$E_x = \frac{-\partial v}{\partial x}, \quad E_y = \frac{-\partial v}{\partial y} \text{ and } E_z = \frac{-\partial v}{\partial z}$$

∴ Equation (ii) becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\rho / \varepsilon_0 k \quad \text{——— (VII)}$$

and equation (4) becomes

$$\text{div grad } v = \partial^2 v = -\rho\,\varepsilon_0 k \quad \text{—— (VIII)}$$

Similarly equation (VII) and (VIII) reduce to

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\rho/\varepsilon_0 \quad \text{—— (IX)}$$

and $\nabla^2 v = -\rho/\varepsilon_0$ —— (X)

four $k = 1$ for air

Equqation (2), (4), (7) and (8) are know as poission equation expressed in different forms. The expre-ssion in the L.H.S of equation (2) is called diver-gence E and hence from equation (4) divergence is defined as the number of flux lines originating per unit volume. The notation ∇ is an operator and is pronounced as 'del" which means differentation.

when $\rho = 0$ i.e when there is no charge in the field equation (2) (4), (7) and (8) becomes

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

or $\nabla \cdot \vec{E} = 0$ ——— (XI)

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

or $\nabla^2 v = v$ ——— (XII)

These are laplace equation.

These equation are found useful in deriving Maxwell's equation for electromagnetic waves.

The End