$\|x\|_\infty$ is a norm on ℓ^∞.

i) $\|x\|_\infty = \sup_{1\le j<\infty} |\xi_j| \geq 0$ (non-negativity)

(ii) $\|x\|_\infty = 0 \Leftrightarrow \sup_{1\le j<\infty} |\xi_j| = 0$

$\Leftrightarrow \xi_j = 0 \;\forall j$

$\Leftrightarrow x = 0$

(iii) $\|\alpha x\|_\infty = \sup_{1\le j<\infty} |\alpha \xi_j|$

$= |\alpha| \sup_{1\le j<\infty} |\xi_j|$

$= |\alpha| \|x\|_\infty$. (homogenity)

(iv) $\|x+y\|_\infty = \sup_{1\le j<\infty} |\xi_j + \eta_j|$ where $x = \langle \xi_j \rangle_{j\ge1}^\infty, y = \langle \eta_j \rangle_{j\ge1}^\infty \in \ell^\infty$

$\le \sup_{1\le j<\infty} (|\xi_j| + |\eta_j|)$

$\le \sup_{1\le j<\infty} |\xi_j| + \sup_{1\le j<\infty} |\eta_j|$

$= \|x\|_\infty + \|y\|_\infty$ (tringle ineq.)

$\left\{ \begin{array}{l} \because |\xi_j| \le \sup_{1\le j<\infty} |\xi_j| \\ |\eta_j| \le \sup_{1\le j<\infty} |\eta_j| \\ \Rightarrow |\xi_j| + |\eta_j| \le \sup|\xi_j| + \sup|\eta_j| \\ \Rightarrow \sup(|\xi_j| + |\eta_j|) \le (\;\text{"}\;) \end{array} \right.$

$\therefore (\ell^\infty, \|\cdot\|_\infty)$ is a Banach space (Normspace)

(7) **The sequence space C_0:-**

$C_0 = \{ x = \langle \xi_j \rangle_{j\ge1}^\infty \mid \langle \xi_j \rangle_{j\ge1}^\infty \longrightarrow 0 \}$

Define $\|\cdot\|_\infty : C_0 \longrightarrow \mathbb{R}$ by

$\|x\|_\infty = \sup_{1\le j<\infty} |\xi_j| \;\forall\; x = \langle \xi_j \rangle \in C_0$.

$(C_0, \|\cdot\|_\infty)$ is a normed space, $C_0 \subset \ell^\infty$ ($\because$ every cgt seq. is bdd)

(8) **The sequence space C:-**

$C = \{ x = \langle \xi_j \rangle : \langle \xi_j \rangle_{j\ge1}^\infty$ is convergent $\}$.

$\|x\|_\infty = \sup_{1\le j<\infty} |\xi_j| \;\forall\; x = \langle \xi_j \rangle \in C$

$(C, \|\cdot\|_\infty)$ is normed space $C_0 \subset C \subset \ell^\infty$ [every cgt seq. is bdd]