$\Rightarrow$ for each $\epsilon>0$, $\exists$ a +ve integer N s.t.

$\|x_n - x\|_\infty < \epsilon/2 \quad \forall\ n \geq N.$

$\Rightarrow \|\xi_j^n - \xi_j\| < \epsilon/2 \quad \forall\ n \geq N$ & $\forall j \in \mathbb{N}$ —— ①

In general,

$|\xi_j^N - \xi_j| < \epsilon/2 \quad \forall\ j \in \mathbb{N}$ —— ②

now,

$x_N = \langle \xi_j^N \rangle \in c_0$ (by (*))

$\Rightarrow \langle \xi_j^N \rangle \longrightarrow 0$

$\Rightarrow$ for above $\epsilon>0$, $\exists$ a +ve integer K s.t. $|\xi_j^N| < \epsilon/2 \quad \forall\ j \geq K.$ —— ③

Consider,

$$|\xi_j| = |\xi_j - \xi_j^N + \xi_j^N|$$
$$\leq |\xi_j^N - \xi_j| + |\xi_j^N|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall\ j \geq K.$$

$\Rightarrow |\xi_j| < \epsilon$ (ϵ g t)

$\Rightarrow \langle \xi_j \rangle \longrightarrow 0.$

$\Rightarrow x = \langle \xi_j \rangle \in c_0.$

$\Rightarrow \overline{c_0} \subset c_0$

$\Rightarrow c_0 = \overline{c_0}$

$\Rightarrow c_0$ is closed in ℓ^∞

$\Rightarrow c_0$ is complete.

$\Rightarrow (c_0, \|\cdot\|_\infty)$ is a Banach space.

⑥ The function space $(C[a,b], \|\cdot\|_\infty)$ is complete where

$$\|x\|_\infty = \sup_{t\in J} |x(t)| \qquad , J = [a,b]$$

Let $\langle x_n \rangle$ be any Cauchy sequence in $C[a,b]$. Then for any $\epsilon>0$, $\exists$ a +ve integer N s.t.

$$d_\infty(x_n, x_m) = \sup_{t\in J} |x_n(t) - x_m(t)| < \epsilon \quad \forall\ n,m \geq N \text{ —— ①}$$

$\Rightarrow |x_n(t) - x_m(t)| < \epsilon \quad \forall\ n,m \geq N$ & $\forall\ t \in [a,b]$