$x_n \longrightarrow x$ uniformly.
& If x_n is cts then x is also cts.

Hence for each fixed $t = t_0 \in J$, we have

$$|x_n(t_0) - x_m(t_0)| < \epsilon \quad \forall\ n, m \geq N.$$

$\Rightarrow \langle x_n(t_0) \rangle$ is a Cauchy sequence in $\mathbb{R}$
Since $\mathbb{R}$ is complete
$\Rightarrow \langle x_n(t_0) \rangle$ is convergent in $\mathbb{R}$
Let $\lim_{n \to \infty} x_n(t_0) = x(t_0) \in \mathbb{R}$

Thus, for each $t \in J$, we have unique real number $x(t)$
This defines (pointwise) a function x on J
Letting $m \to \infty$ in (1)

$$\sup_{t \in J} |x_n(t) - x(t)| < \epsilon \quad \forall\ n \geq N$$

$$\Rightarrow |x_n(t) - x(t)| < \epsilon \text{ unif. on } J = [a,b]$$

$\because x_n$'s are cts on J & the convergence is uniform
$\Rightarrow$ the limit function x is conts on J
Hence $x \in C[a,b]$
Also, $x_n \longrightarrow x$
$\Rightarrow C[a,b]$ is complete
$\Rightarrow (C[a,b], \|\cdot\|_\infty)$ is a Banach space.

(3)
(7) The function space $(C[a,b], \|\cdot\|_1)$ is incomplete
where

$$\|x\|_1 = \int_a^b |x(t)|\,dt \quad , t \in J = [a,b]$$

Sol^n: Let $\langle x_n \rangle$ be any Cauchy sequence in $C[0,1]$
Define

$$x_n(t) = \begin{cases} n & ; \ 0 \leq t \leq \frac{1}{n^2} \\ \frac{1}{\sqrt{t}} & ; \ \frac{1}{n^2} < t \leq 1 \end{cases}$$

Let $\epsilon > 0$ be arbitrary.
Then $\forall\ n > m$, we have