2.3 Further properties of Normed spaces.

By definition,
A subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on X. If Y is closed then Y is called a closed subspace of X.

By definition,
A subspace Y of a Banach space X is a subspace of X considered as a normed space. Hence we do not require Y to be complete.

(2.3-1 Theorem). A subspace Y of a Banach space X is complete iff the set Y is closed in X.

proof: Let Y be closed in X.
Let $\langle y_n \rangle$ be any Cauchy sequence in Y.
since $Y \subset X$.
$\Rightarrow \langle y_n \rangle$ is Cauchy sequence in X.
$\Rightarrow \langle y_n \rangle \longrightarrow y \in X$ ($\because X$ is complete)
since $\langle y_n \rangle \subset Y$ & $\langle y_n \rangle \longrightarrow y$
$\Rightarrow y$ is a limit point of $\langle y_n \rangle$ in Y
$\Rightarrow y \in \overline{Y}$.
but Y is closed (i.e $Y = \overline{Y}$)
$\Rightarrow y \in Y$.
i.e, $\langle y_n \rangle \longrightarrow y \in Y$.
$\Rightarrow Y$ is complete.
Conversely,
Let Y be a complete subspace of X.
T.S: Y is closed in X.
Let $x \in \overline{Y}$.
$\Rightarrow \exists$ a sequence $\langle x_n \rangle$ in Y s.t. $x_n \rightarrow x$.
$\Rightarrow \langle x_n \rangle$ is a Cauchy $\Big\{ \because$ Every cgt sequence in a B.space is Cauchy
and Y is complete