Question 1

Evaluate $\int (x^2 / \sqrt{(1 - x^2)})\, dx$

Solution:

We can write $x^2 = (1 - (1 - x^2))$.

So, $I = \int (1/\sqrt{(1 - x^2)})\, dx - \int \sqrt{(1 - x^2)}\, dx$

We know: $\int (1/\sqrt{(1 - x^2)})\, dx = \sin^{-1}x$, and $\int \sqrt{(1 - x^2)}\, dx = ½(x\sqrt{(1 - x^2)} + \sin^{-1}x)$

Therefore, $I = \sin^{-1}x - ½(x\sqrt{(1 - x^2)} + \sin^{-1}x)$

Simplify: $I = ½(\sin^{-1}x - x\sqrt{(1 - x^2)}) + C$

Answer: $I = ½(\sin^{-1}x - x\sqrt{(1 - x^2)}) + C$

Question 2

Find $\int dx / ((x + 1)\sqrt{(x^2 - 1)})$

Solution:

Let $x = \sec\theta \Rightarrow dx = \sec\theta \tan\theta \, d\theta$ and $\sqrt{(x^2 - 1)} = \tan\theta$.

Then $I = \int (\sec\theta \tan\theta) / ((\sec\theta + 1)\tan\theta) \, d\theta = \int \sec\theta / (\sec\theta + 1) \, d\theta$.

Multiply numerator and denominator by $(\sec\theta - 1)$:

$I = \int [\sec\theta(\sec\theta - 1)] / (\tan^2\theta) \, d\theta = \int (\sec^2\theta - \sec\theta) / \tan^2\theta \, d\theta$.

We know: $\int \csc^2\theta \, d\theta = -\cot\theta$ and $\int (\sec\theta / \tan^2\theta) \, d\theta = -\sec\theta / \tan\theta$.

So, $I = -\cot\theta + \sec\theta/\tan\theta + C$.

Back-substitute $x = \sec\theta \Rightarrow \cot\theta = \sqrt{(x^2 - 1)}/x$, $\sec\theta/\tan\theta = x/\sqrt{(x^2 - 1)}$.

Thus, $I = -\sqrt{(x^2 - 1)}/x + x/\sqrt{(x^2 - 1)} + C = 1 / (x\sqrt{(x^2 - 1)}) + C$.

Answer: $I = 1 / (x\sqrt{(x^2 - 1)}) + C$

Question 3

Evaluate $\int x^2 e^x \sin x \, dx$

Solution:

Let $I = \int x^2 e^x \sin x \, dx$.

Take $u = x^2$, $dv = e^x \sin x \, dx \Rightarrow du = 2x \, dx$, $v = ½ e^x (\sin x - \cos x)$.

So, $I = ½x^2 e^x (\sin x - \cos x) - \int e^x (\sin x - \cos x) x \, dx$.

Now use parts again on $\int x e^x (\sin x - \cos x) \, dx$.

Let $u = x$, $dv = e^x (\sin x - \cos x) \, dx \Rightarrow du = dx$, $v = e^x \sin x$.

So, $\int x e^x (\sin x - \cos x) \, dx = x e^x \sin x - \int e^x \sin x \, dx = x e^x \sin x - ½ e^x (\sin x - \cos x)$.

Hence, $I = ½x^2 e^x (\sin x - \cos x) - [x e^x \sin x - ½ e^x (\sin x - \cos x)]$.

Simplify: $I = e^x [½(x^2 + 1)(\sin x - \cos x) - x \sin x] + C$.

Answer: $I = e^x [½(x^2 + 1)(\sin x - \cos x) - x \sin x] + C$