Page No.
Date

8(8) If $S \subseteq T \subseteq \mathbb{R}$, where $S \neq \phi$, then show that (i) if T is bounded above, then
$$\sup S \leq \sup T.$$
(ii) if T is bounded below, then
$$\inf T \leq \inf S.$$

Solⁿ-(i) $\because$ T is bounded above

$\therefore$ Let $\alpha = \sup T$, then by defⁿ,
$$y \leq \alpha \quad \forall\, y \in T.$$
Since $S \subseteq T$; $x \leq \alpha \;\forall\, x \in S$.
It follows that α is an upper bound of S and so $\sup S \leq \alpha$.
Hence $\sup S \leq \sup T$
#

(ii) give T is bounded below.
Let $\beta = \inf T$
$\therefore$ By defⁿ, $\beta \leq y$, $\forall\, y \in T$
Since $S \subseteq T$; $\beta \leq x \;\forall\, x \in S$.
It follows that β is lower bound of S and so $\beta \leq \inf S$.
Hence $\inf T \leq \inf S$
#.

4. COMPLETENESS IN THE SET OF REAL NUMBERS

* Properties of $\mathbb{R}$ as a Complete order Field:-
The set $\mathbb{R}$ of real numbers satisfies the following 16 properties:-

A_1: $a \in \mathbb{R}$ and $b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$.

A_2: $a+b = b+a \;\forall\, a, b \in \mathbb{R}$.

A_3: $(a+b)+c = a+(b+c) \;\forall\, a, b, c \in \mathbb{R}$

A_4: $a+0 = a \;\forall\, a \in \mathbb{R}$

A_5; To each $a \in \mathbb{R}$, $\exists$ $-a \in \mathbb{R}$ s.t. $a+(-a)=0$

M_1; $a \in \mathbb{R}$ and $b \in \mathbb{R} \Rightarrow ab \in \mathbb{R}$.
M_2; $ab = ba \quad \forall\, a, b \in \mathbb{R}$.
M_3; $(ab)c = a(bc) \quad \forall\, a, b, c \in \mathbb{R}$.
M_4; $a \cdot 1 = a \quad \forall\, a \in \mathbb{R}$
M_5; To each $0 \neq a \in \mathbb{R}$, $\exists$ $a^{-1} \in \mathbb{R}$ s.t. $aa^{-1} = 1$.

AM; $a(b+c) = ab + ac \quad \forall\, a, b, c \in \mathbb{R}$

O_1 for any $a, b \in \mathbb{R}$, exactly one of the following is true.
(i) $a > b$ (ii) $a = b$ (iii) $b > a$ (Trichotomy law)

O_2; $a > b$ and $b > c \Rightarrow a > c \quad \forall\, a, b, c \in \mathbb{R}$
O_3, $a > b \Rightarrow a + c > b + c \quad \forall\, a, b, c \in \mathbb{R}$
O_4; $a > b$ and $c > 0 \Rightarrow ac > bc$

The order-Completeness (O.C) property of $\mathbb{R}$ state that:

vvi. O.C; Every non-empty set of Real numbers which is bounded above (below), has the supremum (infimum) in $\mathbb{R}$.

$\Rightarrow$ On account of $\mathbb{R}$ satisfying the above 16 properties, we say that.
The set $\mathbb{R}$ of real numbers is a Complete ordered field..

Remark(1) on account of $\mathbb{R}$ satisfying the first 11 properties viz. $A_1 - A_5$, $M_1 - M_5$, A.M we say that $\mathbb{R}$ is a field.
Since $\mathbb{R}$ also satisfies 4 order properties viz $O_1 - O_4$; we say that $\mathbb{R}$ is an ordered field.