Define a sequence $\langle x_n \rangle_{n\geq 1}^{\infty}$ in X by

$$x_n = \langle \xi_j^{(n)} \rangle$$

where $\xi_j^{(n)} = \begin{cases} \frac{1}{n^2} & ; j = n \\ 0 & ; j \neq n. \end{cases}$

i.e. $x_1 = \langle 1, 0, 0, \ldots \rangle$

$x_2 = \langle 0, \frac{1}{2^2}, 0, \ldots \rangle$

$\vdots$

$x_n = \langle 0, 0, \ldots, \frac{1}{n^2}, 0, 0 \ldots \rangle$

$$\sum_{n\geq 1} \|x_n\|_\infty = \|x_1\|_\infty + \|x_2\|_\infty + \cdots + \|x_n\|_\infty + \cdots$$

$$= 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots$$

$$= \sum_n \frac{1}{n^2} \text{ which is cgt by p-test}$$

$\Rightarrow \sum_n \|x_n\|_\infty$ is cgt

$\Rightarrow \sum_n x_n$ is absolutely cgt

But,

$$\sum_n x_n = x_1 + x_2 + \cdots + x_n + \cdots$$

$$= \langle 1, 0, \ldots \rangle + \langle 0, \frac{1}{2^2}, \ldots \rangle + \cdots + \langle 0, 0, \ldots \frac{1}{n^2}, \ldots \rangle + \cdots$$

$$= \langle 1, \frac{1}{2^2}, \ldots, \frac{1}{n^2}, \ldots \rangle$$

$$\longrightarrow \langle 1, \frac{1}{2^2}, \ldots, \frac{1}{n^2}, \ldots \rangle \notin \Phi \text{ as } n \to \infty$$

$\Rightarrow \sum_n x_n$ is not convergent.

Q8) A normed space X is a Banach space iff every absolutely convergent series in X is cgt in X.

(Note:- If X is not complete (B. space) then converse is not true. proof Q7)

Sol:- Assume that X is a Banach space.