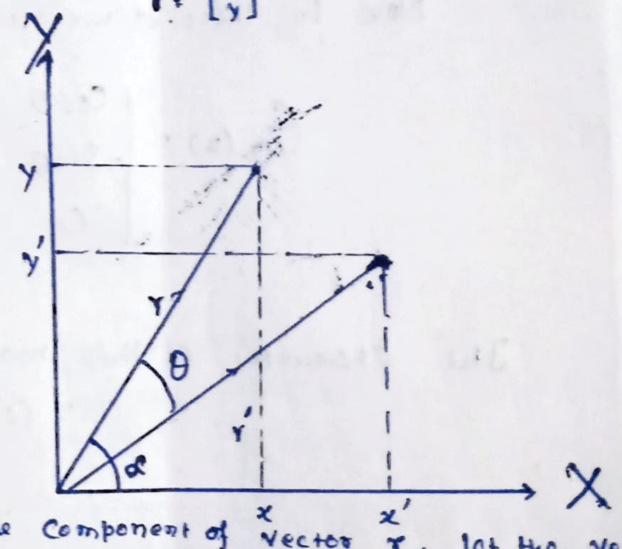
Matrix for Rotation Operation

According to matrix for rotation operation the Z-co-ordinate will be unchanged by any rotation about the Z-axis. Thus, the matrix for rotational operation must be in part as,

$$\begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, finding the four missing elements can be solved as a two dimensional problem in the XY plane.
Let us consider the vector r in the two dimensional co-ordinate system as given in the figure below.
Here the vector r can be expressed as a column matrix r.

$$r = \begin{bmatrix} x \\ y \end{bmatrix}$$



Here x and y are the component of vector r. Let the vector r is rotated clockwise through an angle θ, such that the components of the vector becomes x' and y', and the resulting vector is r'.

$$r' = \begin{bmatrix} x' \\ y' \end{bmatrix} = C_n \cdot r$$

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r(\cos\alpha \cdot \cos\theta + \sin\alpha \cdot \sin\theta) \\ &= r\cos\alpha \cdot \cos\theta + r\sin\alpha \cdot \sin\theta \end{aligned}$$

$$x' = x\cos\theta + y\sin\theta \qquad \text{—— (i)}$$

and

$$\begin{aligned} y' &= r\sin(\alpha - \theta) \\ &= r\sin\alpha \cdot \cos\theta - r\cos\alpha \cdot \sin\theta \\ &= y\cos\theta - x\sin\theta \end{aligned}$$

$$y' = y\cos\theta - x\sin\theta \qquad \text{—— (ii)}$$

These equations (i) and (ii) are represented in the matrix form as.

$$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

This equation is just like

$$r' = C_n . r$$

$$\therefore C_n = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

C_n represents the matrix for rotation operation in two dimensions.

Now in three dimension, the Matrix $C_n(z)$ becomes

$$C_n(z) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The character of this matrix

$$= \cos\theta + \cos\theta + 1 = 2\cos\theta + 1$$

Matrix for Reflection Operation.

To understand Matrix for reflection operation, let us Consider the vector r with Component x and y. By reflection across the plane yz, the components x and y become x' and y'.

The New Component x' and y' are related as

$$x' = 1x + 0y$$
$$y' = 0x + 1y$$

Matrix for this is as follows

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

The reflection operation σ_{yz} is expressed as

$$r' = \sigma_{yz}$$

$$\sigma_{yz} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

In three dimension

$$\sigma_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\sigma_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Character of reflection operator $\sigma = 1+1-1 = 1$

Matrix for Inversion Operation (i)

To know the Matrix inversion operation i, let us consider the x, y and z components of a vector r are transformed into their respective negative by the inversion operation.

$$i.x = -1x + 0y + 0z$$

$$i.y = 0x - 1y + 0z$$

$$i.z = 0x + 0y - 1z$$

$$\therefore \; i \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Here

The Matrix for inversion operation is as follow

$$i = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Character for inversion operation $= -1 -1 -1 = -3$

Matrix for Improper Rotation (S_n)

Now improper rotational axis will be obtained by rotation axis multiplied by σ_h

$$\therefore \; S_n = C_n \, , \sigma_{xy} \cdot (\sigma_h)$$

i.e Matrix of C_n x Matrix of σ_{xy}

$$\therefore \; S_n = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \; S_n = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Character of the Matrix for improper rotation $= \cos\theta + \cos\theta - 1$
$= 2\cos\theta - 1$