* Derive the law of variation of mass with velocity. Explain it's significance.

For deriving the law of variation of mass with velocity expressed by

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \ldots\ldots\ldots (A)$$

we consider elastic collision of two bodies of equal mass. These are symbolised $m_1$ and $m_2$ for convenion. convenience i.e. $m_1 = m_2$ when static.

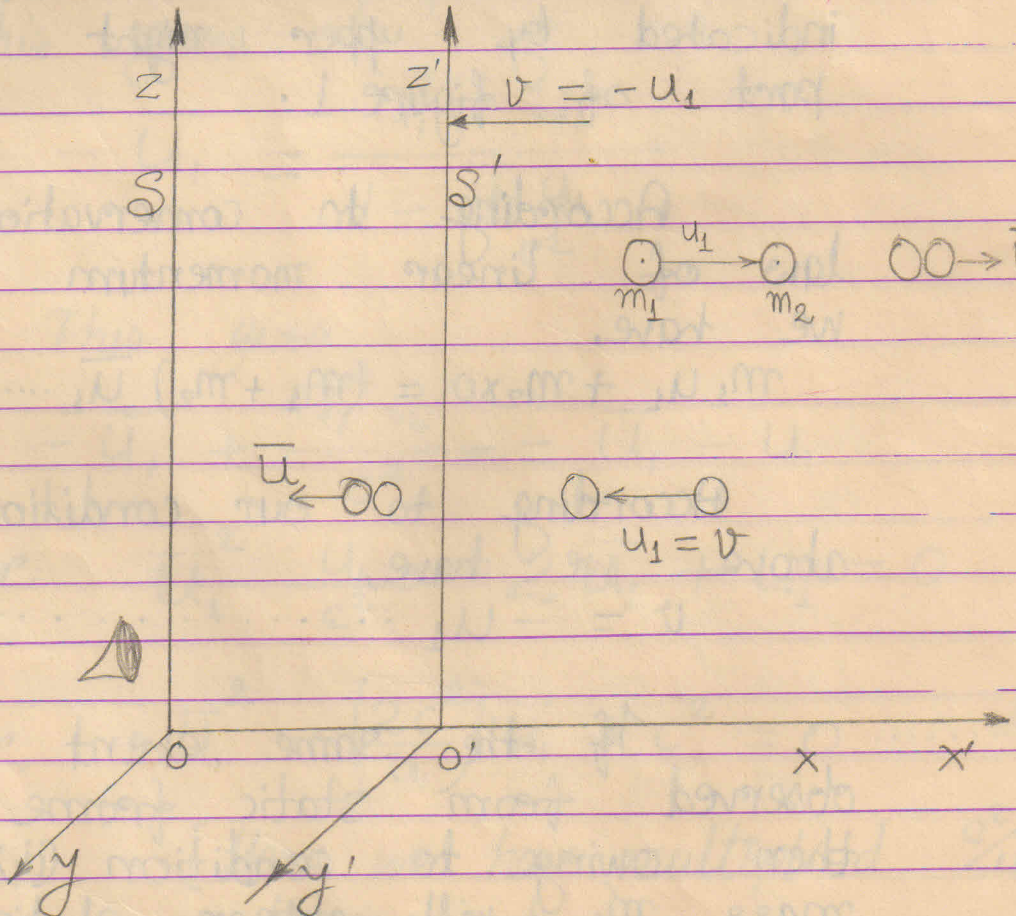


Figure-1 In static position of S and moving S' equal masses $m_1$ and $m_2$ collide elasticaly.

In static position, $m_1 = m_2$.
In figure 1, the moving frame S' is supposed to have velocity $-v$ relative to S. In this (i.e., S') mass $m_1$ is supposed to move to the right with velocity $u_1 = -v$ as indicated. Mass $m_2$ is static in S'. Therefore $m_1$ hits $m_2$ and combined mass moves with velocity $\overline{u_1}$ to the right as indicated. The sequence of events of this collision observed in S' is indicated by upper right part of figure 1.

According to conservation law of linear momentum we have,

$$m_1 u_1 + m_2 \times 0 = (m_1 + m_2)\,\overline{u_1} \quad \ldots\ldots (i)$$

According to our conditions above we have,

$$v = -u_1 \quad \ldots\ldots\ldots\ldots (ii)$$

If the same event is observed from static frame then owing to condition (ii) mass $m_1$ will appear static while $m_2$ moving to the left with velocity $u_1$ colliding with $m_1$. This is shown by lower left part of figure 1, where joint mass has been indicated to move with velocity $-\overline{u}$.

Since the two masses are equal, initialy we must have symmetrical collision and

$$-\overline{u}_1 = \overline{u} \quad \cdots\cdots\cdots \text{(iii)}$$

According to relativistic law of addition of velocities we have,

$$\overline{u} = \frac{\overline{u}_1 + v}{1 + \frac{\overline{u}_1 v}{c^2}} \quad \cdots\cdots\cdots \text{(iv)}$$

According to our conditions above in equation (ii) and (iii) this gives

$$-\overline{u}_1 = \frac{\overline{u}_1 - u_1}{1 - \frac{\overline{u}_1 u_1}{c^2}}$$

This gives,

$$-\overline{u}_1 + \frac{\overline{u}_1^2 u_1}{c^2} = \overline{u}_1 - u_1$$

or, $$\overline{u}_1^2 \frac{u_1}{c^2} - 2\overline{u}_1 + u_1 = 0$$

or, $$+\overline{u}_1^2 - \left(\frac{2c^2}{u_1}\right)\overline{u}_1 + c^2 = 0 \quad \cdots\cdots\cdots \text{(iva)}$$

[Here we have multiplied $c^2/u_1$ throughout]

This is a quadratic equation with roots say,

$$\overline{u}_1 = +\frac{c^2}{u_1} \pm \sqrt{\left(\frac{c^2}{u_1}\right)^2 - c^2}$$

or, $$\overline{u}_1 = \frac{c^2}{u_1}\left[1 \pm \sqrt{1-\frac{u_1^2}{c^2}}\right] \quad \ldots\ldots (ivb)$$

In this expression the positive sign before root is physicaly absurd [since $\frac{c^2}{u_1} \nless \overline{u}_1$] therefore we are left with,

$$\overline{u}_1 = \frac{c^2}{u_1}\left[1 - \sqrt{1-\frac{u_1^2}{c^2}}\right] \quad \ldots\ldots (v)$$

For the observer in static frame S mass $m_1$ appears to be static while $m_2$ moving to the left collides with $m_1$ with velocity

$$-u_1 = -v$$

or $$u_1 = v \quad \ldots\ldots\ldots (va)$$

As indicated in figure the joint mass appears to be moving to the left in static frame S ~~from~~ with velocity $\overline{u}$. The conservation law of momentum gives,

$$(m_1 + m_2)\,\overline{u} = m_1 \times 0 + m_2(-u_1) \quad \ldots\ldots\ldots (vb)$$

By use of equation (iii) this gives the result

$$-\overline{u}_1 \times (m_1 + m_2) = -m_2 u_1$$

$$\overline{u}_1 \times (m_1+m_2) = m_2 u_1 \quad \cdots\cdots \quad (vi)$$

In this we substitute the value of $\overline{u}_1$ from equation (v) and obtain,

$$\frac{c^2}{u_1}\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}(m_1+m_2) = m_2 u_1$$

or, $$m_2 u_1 - m_2 \frac{c^2}{u_1}\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}$$
$$= m_1 \cdot \frac{c^2}{u_1}\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}$$

or, $$m_2 c^2\left[\frac{u_1}{c^2} - \frac{1}{u_1}\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}\right]$$
$$= m_1 \cdot \frac{c^2}{u_1}\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}$$

or, $$m_2\left\{\frac{u_1^2}{c^2} - 1 + \sqrt{1-\frac{u_1^2}{c^2}}\right\}$$
$$= m_1\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}$$

or, $$m_2\left[\sqrt{1-\frac{u_1^2}{c^2}} - \left(1-\frac{u_1^2}{c^2}\right)\right]$$
$$= m_1\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}$$

or, $$m_2\sqrt{1-\frac{u_1^2}{c^2}}\,\cancel{\left[1-\sqrt{1-\frac{u_1^2}{c^2}}\right]}$$
$$= m_1\cancel{\left\{1-\sqrt{1-\frac{u_1^2}{c^2}}\right\}}$$

or, $m_2\sqrt{1-\frac{u_1^2}{c^2}} = m_1$

or $m_2 = \frac{m_1}{\sqrt{1-\frac{u_1^2}{c^2}}}$ ......(vii)

As observe from static frame, mass $m_2$ appears to be moving with velocity,

$-v = -u_1$ ...... (v) rewritten

while, $m_1$ appears to be static.

Applying this condition and $m_1 = m_0$ (static mass)

and $m_2 = m$ (moving mass)

equation (vii) gives

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}} \text{......(viii)}$$

This is the law of variation of mass with velocity.

Significance of variation of mass with velocity :-

For ordinary life $v << c$

Thus $\frac{v^2}{c^2} \longrightarrow 0$
we have
$$m = m_0 \quad \ldots\ldots\ldots (ix)$$

This is not observed on sub-atomic scale, i.e; for electron or cathode rays $v \simeq c$
Thus $m > m_0$

In case $v<c$, equation (vii) valid and we obtain
$$m > m_0$$
If $v = c$
then, $m = \frac{m_0}{0}$
$$= \infty$$

The mass becomes infinite. This is not physicaly true and finally
if $v > c$
then $m$ becomes a complex quantity. This is also physicaly absurd.

Finally, we come to the conclusion that, material particle can not have its velocity equal to or greater than that of light. It must always be less than velocity of light ie, $v < c$.