Beta and Gamma Functions:

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$n!$

$\lfloor n \rfloor$

$\frac{1}{\infty} - \frac{1}{e^0}$

1. Prove that $\Gamma 1 = 1$

$$\Gamma 1 = \int_0^\infty e^{-x} x^{1-1} \, dx = \int_0^\infty e^{-x} \, dx = -e^{-x} \Big|_0^\infty$$

$$= -\left[\frac{1}{\infty} - 1\right] = 1$$

2. Prove that (i) $\Gamma(n+1) = n\Gamma(n)$ (ii) $\Gamma(n+1) = n!$

$\int \mathrm{I}\cdot\mathrm{II}$

$\mathrm{I}\int \mathrm{II} - \int d\mathrm{I}\int \mathrm{II}$

(i)

$$\underline{\Gamma n} = \int_0^\infty \underset{\mathrm{II}}{e^{-x}}\, \underset{\mathrm{I}}{x^{n-1}}\, dx$$

$$= \left(\cancel{x^{n-1}\,\frac{e^{-x}}{-1}}\right)\Bigg|_0^\infty - \int_0^\infty (n-1)\,x^{n-2}\cdot\frac{e^{-x}}{-1}\,dx$$

$$= (n-1)\int_0^\infty x^{n-2}\,e^{-x}\,dx = (n-1)\,\Gamma(n-1)$$

$\Gamma n = (n-1)\,\Gamma(n-1)$ or $n \longrightarrow n+1$

$\Gamma(n+1) = n\,\Gamma n$ ✓

(2)

$$\Gamma(n+1) = \int_0^\infty \underset{\mathrm{II}}{e^{-x}} \, \underset{\mathrm{I}}{x^n} \, dx$$

$$= \cancel{x^n} \frac{e^{-x}}{-1}\Big|_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx$$

$$= n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n \cdot (n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\cdots\Gamma(1) = n!$$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n-2) = (n-3)\Gamma(n-3)$

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\Gamma(n-2)$

Q: Evaluate $\Gamma\left(\frac{-5}{2}\right)$

$n = (n-1)\,\Gamma(n-1)$

Soln:

$$\Gamma\left(\frac{-1}{2}+1\right) = \frac{-1}{2}\,\Gamma\left(\frac{-1}{2}\right)$$

$n+1 \qquad n \;\Gamma n$

$\frac{-5}{2} -$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$1 - \frac{1}{2} = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \frac{-1}{2}\,\Gamma\left(\frac{-3}{2}+1\right)$$

$$= -\frac{1}{2}\left(\frac{-3}{2}\right)\Gamma\left(\frac{-3}{2}\right)$$

$$= \frac{-1}{2}\left(\frac{-3}{2}\right)\Gamma\left(1-\frac{5}{2}\right) = \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\Gamma\left(\frac{-5}{2}\right)$$

$$\Gamma\frac{1}{2} = \frac{-15}{8}\,\Gamma\frac{-5}{2}$$

or $$\Gamma\frac{-5}{2} = \frac{-8\sqrt{\pi}}{15}$$

$\Gamma n \;\; x^{n-1}$

$n-1 = \frac{3}{2}$

$n = \frac{5}{2}$

Q. Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx = I$

$(x^{1/2}) = t^2$

$x^{1/4} = (t)^{1/2}$

let $\sqrt{x} = t$; $t^2 = x$

$2t\,dt = dx$

$$\therefore I = 2\int_0^\infty t^{1/2} e^{-t} t\, dt = 2\int_0^\infty t^{3/2} e^{-t}\, dt = 2\int_0^\infty t^{5/2 - 1} e^{-t}\, dt$$

$$= 2\Gamma\left(\tfrac{5}{2}\right) = 2\cdot\tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \cancel{2}\times\tfrac{3}{\cancel{2}}\times\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\tfrac{5}{2} - 1 =$$

$$= \tfrac{3}{2}\sqrt{\pi}$$

H.W. 1. Evaluate $\int_0^\infty \sqrt{x}\, e^{-3\sqrt{x}}\, dx$

2. Evaluate $\int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$

e^{-x}

$+3\sqrt{x} = t$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$