Leibnit'z Theorem:

$n! = n(n-1)(n-2)!\,\cancel{(n-3)!}$

$^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$

$^nC_r = \frac{n!}{r!(n-r)!}$

If u & v are functions of x. Then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0\, u_n\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \dots + {}^nC_n\, u \cdot v_n$$

$$= u_n v + n\, u_{n-1} v_1 + \frac{n(n-1)}{2}$$

$^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} = \frac{n(n-1)}{2}$

Proof: By mathematical Induction:

$n = 1$ ✓

$\boxed{^nC_0 = \frac{\cancel{n!}}{0!\cancel{n!}} = 1}$

$n = k$ (True)

$\downarrow$

$n = k+1$ (we prove)

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = uv_1 + vu_1$$

$${}^2C_1 = \frac{2!}{1!\,(2-1)!} = \frac{2!}{1!\,1!} = 2$$

$$\frac{d^2}{dx^2}(uv) = \frac{d}{dx}(uv_1 + vu_1)$$

$$= u_1v_1 + uv_2 + v_1u_1 + vu_2$$

$$= vu_2 + 2u_1v_1 + uv_2$$

$$= vu_2 + {}^2C_1\, u_1v_1 + uv_2$$

H.W. { Let it be true for $n=k$, $\longrightarrow$ Prove it for $n=k+1$ }

Q: If $y = \cos(m \sin^{-1} x)$. Prove that

$$(1-x^2)\, y_{n+2} - (2n+1)\, y_{n+1} \cdot x + (m^2 - n^2)\, y_n = 0$$

Solu^n:

$$y = \cos(m \sin^{-1} x)$$

$$\frac{dy}{dx} = y_1$$

$$\frac{dy}{dx} = y_1 = -\sin(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

or $\sqrt{1-x^2}\, y_1 = -m \sin(m \sin^{-1} x)$.

squaring both sides.

$$(1-x^2)\, y_1^2 = m^2 \sin^2(m \sin^{-1} x)$$

$u \quad v$

$$u\frac{dv}{dx} + \frac{du}{dx}\cdot v \qquad = m^2\left[1 - \underbrace{\cos^2(m\sin^{-1}x)}\right]$$

$$= m^2\left[1 - y^2\right]$$

Diff. again we get,

$$\cancel{2}\underline{(1-x^2)}\, y_1 \frac{dy_1}{dx} - \cancel{2}x\, y_1^2 = m^2(-\cancel{2}y \cdot y_1)$$

$$(1-x^2)y_1 y_2 - x y_1^2 + m^2 y y_1 = 0$$

$\cos^2 x + \sin^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\frac{dy}{dx} = y_1$

$\frac{d^2y}{dx^2} = y_2$

$\vdots$

$\frac{d}{dx}(uv)$

$= u\frac{dv}{dx} + v\frac{du}{dx}$

$$y_1\left[(1-x^2)y_2 - xy_1 + m^2 y\right] = 0$$

$$\frac{d}{dx^n}(x y_1) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} \ldots$$
$$= 1\cdot y_{n+1}\cdot x + n\, y_n \cdot 1$$

Applying Leibnitz Th. on $\underbrace{(1-x^2)}_{v}\underbrace{y_2}_{u} - xy_1 + m^2 y = 0$

$(1-x^2)\cdot y_2$

$${}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$$

$v_1 = -2x$, $v_2 = -2$

$v_3 = 0$

$${}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1$$

$$\cancel{\text{or}}\; y_{n+2}(1-x^2) + n(-2x)\,y_{n+1} - \cancel{2}\frac{n(n-1)}{\cancel{2}}\,y_n - xy_{n+1} - ny_n$$

$$+ m^2 y_n = 0$$

$$-n(n-1) - n + m^2$$
$$-n^2 + \cancel{n} - \cancel{n} + m^2$$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

<u>H.W.</u> :→ If $y = \sin(m\sin^{-1}x)$. Prove that

$$(1-x^2)y_{n+2} - (2n+1)\cdot x\cdot y_{n+1} + (m^2-n^2)y_n = 0$$

<u>Q</u>: Find n^{th} derivative of $x^{n-1}\log x$.

$$y = x^{n-1}\log x$$

$$y_1 = (n-1)x^{n-2}\log x + \frac{x^{n-1}}{x}$$

or $xy_1 = (n-1)x^{n-1}\log x + x^{n-1}$

$\frac{d}{dx}(uv)$
$= u\frac{dv}{dx} +$
$v\frac{du}{dx}$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

H.W. :→ If $y = \sin(m\sin^{-1}x)$. Prove that
$$(1-x^2)y_{n+2} - (2n+1)\cdot x\cdot y_{n+1} + (m^2-n^2)y_n = 0$$

Q: Find n^{th} derivative of $x^{n-1}\log x$.

$$y = x^{n-1}\log x$$

$$y_1 = (n-1)x^{n-2}\log x + \frac{x^{n-1}}{x}$$

or $xy_1 = (n-1)x^{n-1}\log x + x^{n-1}$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$xy_1 = (n-1)y + x^{n-1}$$

$$\frac{d}{dx^{n-1}}\left(\frac{dy}{dx}\right)$$

$$\frac{d}{dx^3}x^3 = 3!$$

Applying Leibnitz Th. (n-1) times

$$\frac{d}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots$$

$$\frac{d}{dx^{n-1}}(uv) = {}^{n-1}C_0 u_{n-1} v + {}^{n-1}C_1 u_{n-2} v_1 + {}^{n-1}C_2 u_{n-3} v_2 + \dots$$

$$\therefore\ xy_n + (n-1)\cancel{y_{n-1}}\cdot 1 = (n-1)\cancel{y_{n-1}} + (n-1)!$$

$$\text{or } xy_n = (n-1)! \quad \text{or } \boxed{y_n = \frac{(n-1)!}{x}}$$