The n^{th} derivative of a function for $x=0$

Working Rule:

1. Write down the given fn. as $y=f(x)$.
2. Find y_1 3. Find y_2
4. Diff. n times both the sides of the eqn obtained in step 3 by Leibnitz Rule.
5. Substitute $x=0$ in y, y_1, y_2 and n^{th} derivative
6. Put the values of $y(0), y_1(0), y_2(0)$ in step 5.

7. Put $n = 1, 2, 3, 4$ in last eqn of 6.

8. Find $y_n(0)$ when n is even and n is odd.

Q: If $y = (\sin^{-1}x)^2$. Prove that $(y_n)_0 = 0$ for n odd

$(y_n)_0 = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \ldots (n-2)^2$, $n \neq 2$, n is even.

Solun:
$$y = (\sin^{-1}x)^2 \quad \text{—} \quad (1)$$

$$y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$$

Squaring

$$(1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y \quad — \quad (2)$$

Diff. (2) w.r.t 'x'

$$2(1-x^2)y_1y_2 - 2xy_1^2 = 4y_1$$

or $(1-x^2)y_1y_2 - xy_1^2 - 2y_1 = 0$

or $y_1[\underbrace{(1-x^2)y_2 - xy_1}\, - 2] = 0 \quad — \quad (3)$

Diff. 'n' times using Leibnitz Th.

$$y_{n+2}(1-x^2) - 2nx\,y_{n+1} - \frac{\cancel{2}n(n-1)}{\cancel{2!}}y_n - xy_{n+1} - ny_n = 0$$

$$(1-x^2)\, y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0 \qquad — (4)$$

Putting $x=0$ in (1), (2), (3) & (4)

$(y)_0 = 0$, $(y_1)_0 = 0$; $(y_2)_0 = 2$

and $(y_{n+2})_0 = n^2 (y_n)_0$

If <u>n is odd</u>: $n = 1, 3, 5 \ldots$

$(y_3)_0 = (y_1)_0 = 0$

$(y_5)_0 = 9 (y_3)_0 = 0 \ldots$ $\qquad (y_n)_0 = 0$ when n is odd.

when n is even, $n = 2, 4, 6 \cdots$

$$(y_4)_0 = 2^2 (y_2)_0 = 2^2 \cdot 2$$
$$= 2 \cdot 2^2$$

$$(y_6)_0 = 4^2 (y_4)_0 = 4^2 \cdot 2^2 \cdot 2$$
$$= 2 \cdot 2^2 \cdot 4^2$$

$\vdots$

Hence, in general,

$$(y_n)_0 = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \cdots (n-2)^2, \quad n \neq 2$$

H.W. If $y = [x + \sqrt{1+x^2}]^m$. Prove that

$$(1+x^2) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0. \quad (y_n)_0 = ?$$