Blackbody Radiation-Section4

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1 Quantum Approach Towards Blackbody Radiation Theory

1.1 Planck's hypothesis

After the failed attempts of Rayleigh-Jeans and Wien's classical theory to explain the blackbody radiation curve, Max Planck succeeded in deriving a formula which agrees extremely well with the experimental results. According to the classical theory of radiation the oscillators (vibrating atoms and molecules on the inner cavity wall) can exchange (absorb and radiate) energy continuously. Planck discarded the idea of continuity in energy transfer. His assumptions are:

(A) Energy of each simple harmonic oscillator inside the blackbody radiation chamber may have any series of discrete values, but not in between i.e. their energies can not have any continuous values.

(B) The oscillators can not radiate energy or absorb energy continuously. However, an oscillator of frequency ν can only radiate or absorb energy in units of $h\nu$, where h is Planck's constant. (h=6.626$\times10^{-34}$J.s) Therefore, if an oscillator of energy E_1 changes its state to energy E_2, then the energy radiated by that oscillator is E_2-E_1=$nh\nu$, where n=1,2,3....*etc.*

Based on these assumptions, Planck invented the *quantum theory of blackbody radiation*, which unlocked the door towards the world of quantum physics.

1.2 Derivation of Planck's law of blackbody radiation

Planck considered that the energy of oscillators have discrete set of values ϵ, 2ϵ, 3ϵ,...$n\epsilon$ *etc.* He considered the Maxwell-Boltzman energy distribution among the oscillators. Therefore, number of oscillators having energy 0 is N_0, number of oscillators having energy ϵ is $N_0 e^{-\epsilon/k_BT}$, number of oscillators with energy 2ϵ is $N_0 e^{-2\epsilon/k_BT}$ and so on. (Here, k_B is the Boltzman constant, $k_B = 1.38\times10^{-23}$ J/K). If the total number of oscillators is N, then we write:

$$N = N_0 + N_0 e^{-\epsilon/k_BT} + N_0 e^{-2\epsilon/k_BT} + ... + N_0 e^{-n\epsilon/k_BT} + ... \tag{1}$$

where, $N_0 e^{-n\epsilon/k_BT}$ is the number of oscillators with energy $n\epsilon$. Putting ϵ/k_BT=x, eqn (1) becomes:

$$N = N_0 + N_0 e^{-x} + N_0 e^{-2x} + N_0 e^{-3x} + ... + N_0 e^{-nx} + ... \tag{2}$$

$$N = \frac{N_0}{1-e^{-x}} \tag{3}$$

If the total energy of all oscillators is E, then:

$$E = 0\times N_0 + \epsilon N_0 e^{-x} + 2\epsilon N_0 e^{-2x} + 3\epsilon N_0 e^{-3x} ... + n\epsilon N_0 e^{-nx} + ... \tag{4}$$

Multiplying e^{-x} on both sides of eqn(4)

$$Ee^{-x} = \epsilon N_0 e^{-2x} + 2\epsilon N_0 e^{-3x} + 3\epsilon N_0 e^{-4x} ... + n\epsilon N_0 e^{-(n+1)x} + ... \tag{5}$$

subtracting eqn(5)from eqn(4)

$$E(1-e^{-x}) = \epsilon N_0 e^{-x} + \epsilon N_0 e^{-2x} + \epsilon N_0 e^{-3x} + ... \tag{6}$$

$$E(1-e^{-x}) = \frac{\epsilon N_0 e^{-x}}{1-e^{-x}} \tag{7}$$

$$E = \frac{\epsilon N_0 e^{-x}}{(1-e^{-x})^2} \tag{8}$$

Using eqn (3) and (8), average energy per oscillator is:

$$\bar{\epsilon} = \frac{E}{N} = \frac{\epsilon e^{-x}}{1-e^{-x}} = \frac{\epsilon}{e^x - 1} \tag{9}$$

According to Planck's hypothesis, the energy quanta associated to each oscillator is $\epsilon = h\nu = \frac{hc}{\lambda}$ (since, $\nu = \frac{c}{\lambda}$). Therefore, $x = \frac{\epsilon}{k_B T} = \frac{hc}{\lambda k_B T}$. Hence,

$$\bar{\epsilon} = \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1} \tag{10}$$

We have seen before, the number of oscillators per unit volume, considering both the horizontal and vertical polarization of radiation, within wavelength range λ to $\lambda + d\lambda$ is: $8\pi\lambda^{-4}d\lambda$. Hence the energy density (u_λ) of radiation within wavelength range λ to $\lambda + d\lambda$ may be expressed as:

$$\frac{\partial u_\lambda}{\partial \lambda} = \frac{hc/\lambda}{e^{hc/\lambda k_B T} - 1} \times \frac{8\pi}{\lambda^4} \tag{11}$$

$$\frac{\partial u_\lambda}{\partial \lambda} = \frac{8\pi hc\lambda^{-5}}{(e^{hc/\lambda k_B T} - 1)} \tag{12}$$

Equation (12) is the *Planck's law of blackbody radiation* expressed in terms of wavelength λ. This equation can be written in terms of frequency ν as follows:

$$\frac{\partial u_\nu}{\partial \nu} = \frac{8\pi h\nu^3}{c^3(e^{\frac{h\nu}{k_B T}} - 1)} \tag{13}$$

Equation (13) is the *Planck's law of blackbody radiation* expressed as a function of frequency ν.

1.3 Derivation of Rayleigh-Jeans' law from Planck's law

Planck's law of blackbody radiation reduces to Rayleigh-Jean's formula at long wavelength limit. For large value of λ,

$$e^{hc/\lambda k_B T} \approx 1 + (\frac{hc}{\lambda k_B T})$$

$$e^{hc/\lambda k_B T} - 1 \approx \frac{hc}{\lambda k_B T}$$

Hence Planck's law (equation 12) reduces to:

$$\frac{\partial u_\lambda}{\partial \lambda} = \frac{8\pi hc\lambda^{-5}}{(hc/\lambda k_B T)} = \frac{8\pi k_B T}{\lambda^4} \tag{14}$$

which is *Rayleigh-Jeans'* formula for blackbody radiation.

1.4 Derivation of Wien's distribution law from Planck's law

At small wavelength limit Planck's law of blackbody radiation reduces to Wien's distribution law. If λ has small value, $e^{hc/\lambda k_B T}$ is large compared to 1,

$$e^{hc/\lambda k_B T} \gg 1$$

$$e^{hc/\lambda k_B T} - 1 \approx e^{hc/\lambda k_B T}$$

Therefore equation (12) reduces to:

$$\frac{\partial u_\lambda}{\partial \lambda} = 8\pi hc\lambda^{-5} e^{-hc/\lambda k_B T} \tag{15}$$

Which is Wien's distribution law for blackbody radiation.

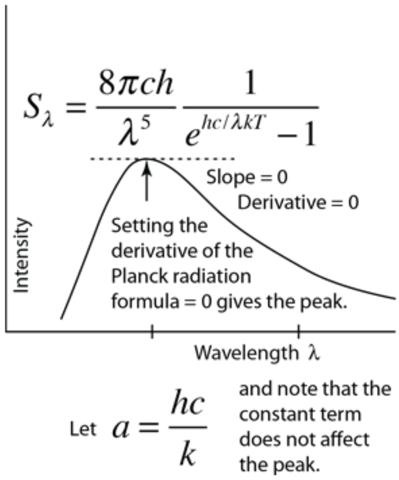

Figure 1:

1.5 Derivation of Wien's displacement law from Planck's law

Equation (12), gives the radiated energy per unit volume per unit wavelength. Let's say:

$$S_\lambda = \frac{\partial u_\lambda}{\partial \lambda}$$

In figure 1, we have shown a characteristic blackbody radiation spectrum at a fixed temperature T. The peak intensity is I_{peak} or I_m. Let's denote the corresponding wavelength as λ_m. Derivative of S_λ with respect to λ gives slope on the curve. The slope is zero at peak intensity I_m. From eqn (12):

$$S_\lambda = \frac{8\pi hc\lambda^{-5}}{(e^{hc/\lambda k_B T} - 1)} \tag{16}$$

lets say,

$$\frac{hc}{k_B} = a(constant)$$

Differentiating S_λ with respect to λ:

$$\frac{dS_\lambda}{d\lambda} = \frac{d}{d\lambda}[\frac{1}{\lambda^5(e^{a/\lambda T} - 1)}] \tag{17}$$

Therefore,

$$\frac{dS_\lambda}{d\lambda} = [\frac{-5}{\lambda^6(e^{a/\lambda T} - 1)} + \frac{1}{\lambda^5}\frac{-e^{(a/\lambda T)}\frac{a}{T}\frac{-1}{\lambda^2}}{(e^{a/\lambda T} - 1)^2}] \tag{18}$$

So,

$$\frac{dS_\lambda}{d\lambda} = \frac{1}{\lambda^5(e^{a/\lambda T} - 1)}[\frac{-5}{\lambda} + \frac{e^{a/\lambda T}\frac{a}{\lambda T^2}}{(e^{a/\lambda T} - 1)}] \tag{19}$$

Equating the slope $\frac{dS_\lambda}{d\lambda}$ to zero, and putting $\lambda = \lambda_m$ we get:

$$e^{a/\lambda_m T}\frac{a}{\lambda_m^2 T} = \frac{5}{\lambda_m}(e^{a/\lambda_m T} - 1) \tag{20}$$

$$\lambda_m T = \frac{ae^{a/\lambda_m T}}{5(e^{a/\lambda_m T} - 1)} = \frac{a}{5(1 - e^{-a/\lambda_m T})} \tag{21}$$

This equation can not be solved analytically, but may be solved numerically which gives: $\lambda_m T$=2.89×10^{-3}m.K. (constant). Which proves Wien's displacement law.

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[1] Figures are collected from online resources.