LINEAR ALGEBRA

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Characteristic matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. λ be any scalar. I be a unit matrix of order n.

$\therefore$ $A - \lambda I$ is the characteristic matrix.

Characteristic equation

$|A - \lambda I| = 0$ is characteristic equation of A.

Characteristic roots / latent roots / eigen values

Roots of $|A - \lambda I| = 0$ are eigen values of A.
ie. different values of λ are eigen values of A.

Eigen vectors

If λ is eigen value of A then $\exists$ a non zero vector X such that

$$A X = \lambda X.$$

X is called eigen vector of A corresponding to eigen value λ.

MATRICES

CAYLEY - HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristics equation.

Proof: Let $A = [a_{ij}]_{n \times n}$ be a square matrix.

i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Let I be a unit matrix of order $n \times n$.

Let λ be a scalar.

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$\Rightarrow |A-\lambda I| = (-1)^n \left[\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \cdots + p_n\right]$

If $|A-\lambda I| = 0$ then

$$\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \cdots + p_n = 0 \quad \text{——} (1)$$

Eqn (1) is the characteristic equation of A.

we've to prove that A satisfies its own characteristic equation.

Let ~~it~~ A satisfies the characteristic eqn.

Putting $\lambda = A$ in (1), we get

$$A^n + p_1 A^{n-1} + p_2 A^{n-2} + \cdots + p_n I = 0$$

where 0 is a zero matrix of order n.

Also, adjoint$(A-\lambda I)$ can be expressed as a matrix polynomial in λ given by

$\text{adj}(A-\lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \cdots + B_{n-1}$ where $B_0, B_1, \ldots, B_{n-1}$ are $n \times n$ matrices whose elements are functions of elements a_{ij}'s.

Since, $A(\text{adjoint})A = |A| I$

$\Rightarrow (A-\lambda I)\ \text{adjoint}(A-\lambda I) = |A-\lambda I| I$

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$$\Rightarrow (A-\lambda I)\left[B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \cdots + B_{n-1}\right]$$

$$= (-1)^n\left[\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n\right]I$$

comparing the coefficients of different powers of λ i.e. $\lambda^n, \lambda^{n-1}, \cdots$, we get

$$-B_0 = (-1)^n I$$

$$AB_0 - IB_1 = (-1)^n p_1 I$$

$$AB_1 - IB_2 = (-1)^n p_2 I$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$AB_{n-1} = (-1)^n p_n I$$

Pre-multiplying the above by $A^n, A^{n-1}, A^{n-2}, \cdots, I$ respectively and adding we get

$$-A^n B_0 + A^{n-1}(AB_0 - IB_1) + A^{n-2}(AB_1 - IB_2) + \cdots$$

$$+ IAB_{n-1} = (-1)^n\left[A^n + p_1 A^{n-1} + p_2 A^{n-2} + \cdots + p_n I\right]$$

$$\Rightarrow A^n + p_1 A^{n-1} + p_2 A^{n-2} + \cdots + p_n I = 0$$

Hence, the matrix A satisfies its own characteristic equation.

Q. (a) Find the eigen values of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Soln: characteristic eqn of A is

$|A - \lambda I| = 0$, where λ is any indeterminate. — (1)

$\Rightarrow A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

From (1) $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 1) = 0 \Rightarrow \lambda = 6, 1$

Hence, eigen values of $A = 6, 1$.

Q. (b) Find the eigen vector of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Soln Eigen values of $A = 6, 1$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector of A

corresponding to the eigen value 6, i.e. $\lambda = 6$.

$\therefore [A - \lambda I] X = 0$

$\Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} X = 0$ but $\lambda = 6$

$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -x_1 + 4x_2 \\ x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 + 4x_2 = 0$ and $x_1 - 4x_2 = 0$

$\Rightarrow x_1 = 4x_2$. If $x_2 = k$ then $x_1 = 4k$.

Thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4k \\ k \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$.

i.e. $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ are some eigen vectors when $k = 1, 2$ respectively, corresponding to $\lambda = 6$.

$\boxed{\text{When } \lambda = 1} \Rightarrow$ $\therefore (A - \lambda I) X = 0 \Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4x_1 + 4x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 + 4x_2 = 0$ and $x_1 + x_2 = 0$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$. When $x_2 = c$ then $x_1 = -c$.

Thus $\begin{bmatrix} -c \\ c \end{bmatrix}$ i.e. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are the eigen vectors corresponding to eigen value $\lambda = 1$.

Q. Find the eigen values of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Let I be a unit matrix of order 3×3.

$$\therefore A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

The characteristic eqn of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(\lambda-2)(\lambda-3)-2\right] - 2\left[2-\lambda-1\right] + \left[2-(3-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-5\lambda+4) + 2(\lambda-1) + (\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)\left[(2-\lambda)(\lambda-4) + 2 + 1\right] = 0$$

$$\Rightarrow (\lambda-1)\left[-(\lambda^2-6\lambda+8)+3\right] = 0$$

$\Rightarrow (\lambda-1)[-\lambda^2+6\lambda-5]=0$

$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+5)=0$

$\Rightarrow (\lambda-1)(\lambda-5)(\lambda-1)=0$

$\Rightarrow \lambda = 1, 1, 5$

Hence, the eigen values of the given matrix $= 1, 1, 5$.

Q. Find the eigen values for the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. Also, verify cayley-Hamilton theorem.

Solution

Here, $A - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}$$

The characteristic eqn of A is

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda(3-\lambda) + 2 = 0 \Rightarrow -3\lambda + \lambda^2 + 2 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$ (1) $\Rightarrow \lambda = 1, 2$

So, the eigen values $= 1, 2$.

Now, to verify cayley-Hamilton theorem, we shall verify, whether A satisfies its characteristic eqn (1) or not.

Putting $\lambda = A$ in (1) $\Rightarrow A^2 - 3A + 2I = 0$

$$LHS = A^2 - 3A + 2I = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ -6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2+2 & 3-3 \\ -6+6 & 7-9+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= 0$ (zero matrix) $=$ RHS. Hence cayley-Hamilton theorem is verified.

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Q. Find the eigen values, eigen vectors ~~and~~ for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Also verify cayley-Hamilton theorem.

Soln $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

characteristic eqn of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 3 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$ (1) $\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda = 5, -1$

So, the eigen values of $A = 5, -1$

Verification of Cayley-Hamilton theorem

we're to prove that $A^2 - 4A - 5I = 0$ [using (1)]

Now, $A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

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$\therefore A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ (Zero matrix)

Thus, cayley Hamilton theorem is verified.

For Eigen vectors Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is eigen vector corresponding to eigen value 5, i.e. $\lambda = 5$.

$\therefore (A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4x_1 + 4x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + 4x_2 = 0$ i.e. $x_1 = x_2$

and $2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$ $\therefore X = \begin{bmatrix} K \\ K \end{bmatrix}$

where K is any constant.

Also, let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be eigen vector corresponding to eigen value $\lambda = -1$

$\therefore (A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x_1 + 4x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x_1 + 4x_2 = 0 \Rightarrow x_1 = -2x_2$.

let $x_2 = k \Rightarrow x_1 = -2k \Rightarrow \begin{bmatrix} -2k \\ k \end{bmatrix}$ is another eigenvector