Linear Transformations

Let $U(F)$ and $V(F)$ be two vector spaces. Let $T: U \to V$ be a mapping. Then T is called a linear transformation if

(i) $T(x+y) = T(x) + T(y) \quad \forall\, x, y \in U$

(ii) $T(Kx) = K\,T(x) \quad \forall\, x \in U,\ K \in F.$

$\boxed{\text{OR}}$

$$T(ax+by) = a\,T(x) + b\,T(y) \quad \forall\, x, y \in U, \quad \forall\, a, b \in F.$$

Q. Let T be the mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $T(a,b) = (a+b, a-b)$.

Prove that T is a linear transformation.

Solution

We shall show that

$$T(x+y) = T(x) + T(y) \text{ and } T(cx) = cT(x).$$

Let $x = (a,b)$, $y = (c,d)$.

$$T(x+y) = T\{(a,b)+(c,d)\} = T\{(a+c),(b+d)\}$$

$$= T(a+c, b+d) = T\{((a+c)+(b+d)), ((a+c)-(b+d))\}$$

$$= T\{((a+b)+(c+d)), ((a-b)+(c-d))\}$$

$$= T\{(a+b, a-b) + (c+d, c-d)\}$$

$$= T(a+b, a-b) + T(c+d, c-d)$$

$$= T(a,b) + T(c,d)$$

$$= T(x) + T(y)$$

$$\Rightarrow T(x+y) = T(x) + T(y)$$

Also, ~~F(ct~~ $T(cx) = T\{c(a,b)\} = T(ca, cb)$

$$= (ca+cb, ca-cb)$$

$$= (c(a+b), c(a-b))$$

$$= C(a+b, a-b)$$

$$= c\,T(a,b) = cT(x)$$

$$\therefore T(cx) = cT(x)$$

$\Rightarrow T$ is linear.

Q: verify whether T is linear when T is defined as $T: R^2 \to R^2$ by $T(x) = 2x$.

Solution

we have to verify that

$T(x+y) = T(x) + T(y)$ and $T(cx) = c[T(x)]$

Given that $T(x) = 2x$ ——(1)

Now $T(U+V) = 2(U+V)$ [using (1)]

$= 2U + 2V$

$\Rightarrow \boxed{T(U+V) = T(U) + T(V)}$ [using (1)] ——(2)

Also, $T(cU) = 2(cU)$ [using (1)]

$= c(2U)$

$= cT(U)$

$\therefore \boxed{T(cU) = cT(U)}$ ——(3)

T satisfies both defining properties, clear from (2) and (3).

$\Rightarrow$ T is linear.

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Q: Prove that $T : R^2 \to R^3$ by ~~the form~~

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-y \\ y \\ x \end{pmatrix} \text{ is linear.}$$

Solution If T is linear then

$$T(U+V) = T(U) + T(V) \quad \text{——— (1)}$$

$$\text{and } T(cU) = cT(U). \quad \text{——— (2)}$$

Given that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-y \\ y \\ x \end{pmatrix}$ — (3)

Let $U = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, V = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

$$T(U+V) = T\left\{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\} = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3(x_1+x_2)-(y_1+y_2) \\ y_1+y_2 \\ x_1+x_2 \end{pmatrix} = \begin{pmatrix} (3x_1-y_1)+(3x_2-y_2) \\ y_1+y_2 \\ x_1+x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x_1-y_1 \\ y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} 3x_2-y_2 \\ y_2 \\ x_2 \end{pmatrix}$$

$$\Rightarrow T(U+V) = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = T(U) + T(V)$$

(1) Proved

Also $T(cU) = T\left\{c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right\} = T\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix}$

$$= \begin{pmatrix} 3(cx_1)-cy_1 \\ cy_1 \\ cx_1 \end{pmatrix} = c\begin{pmatrix} 3x_1-y_1 \\ y_1 \\ x_1 \end{pmatrix} = cT\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\Rightarrow T(cU) = cT(U) \quad \text{— (2) Proved.}$$

Thus, from (1) and (2) we can conclude that T is linear.

Q T is defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |x| \\ y \end{pmatrix}$

Is T linear?

Solution For T to be linear,

$T(U+V) = T(U) + T(V)$ and

$T(c\alpha) = cT(\alpha)$

For 2nd condition, we shall verify that

$$T\left(-\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = -T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$LHS = T\left(-\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} |-1| \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$RHS = -T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} |1| \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$\therefore$ LHS $\neq$ RHS

So, T ~~does~~ is not linear.

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Q. verify that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \end{pmatrix}$ is linear.

Solution For T to be linear, $T(U+V) = T(U)+T(V)$
and $T(c\alpha) = cT(\alpha)$.

let $\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$T\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\times 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$2\,T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1\times 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\therefore T\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \neq 2\,T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e. $T(2\alpha) \neq 2T(\alpha) \Rightarrow T$ is not linear

Q Prove that $T : R^2 \to R^2$ is not linear where

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+1 \\ x-2y \end{pmatrix}$$

Solution

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow T\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\times 0+1 \\ 0-2\times 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ T does not map zero vector to the zero vector

$\Rightarrow$ T is not linear.

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Q: Verify whether $T: R^3 \to R^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}$$ is linear.

Solution For a linear transformation,

$T(x+y) = T(x) + T(y)$ — (1) and

$T(cx) = cT(x)$ — (2).

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\therefore T(X+Y) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(x_1+y_1) + (x_3+y_3) \\ -4(x_2+y_2) \end{bmatrix} = \begin{bmatrix} (2x_1+x_3) + (2y_1+y_3) \\ (-4x_2) + (-4y_2) \end{bmatrix} \text{ — (3)}$$

Also $T(X) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}$

and $T(Y) = T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} 2y_1 + y_3 \\ -4y_2 \end{bmatrix}$

$$\Rightarrow T(X) + T(Y) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_3 \\ -4y_2 \end{bmatrix}$$

$$= \begin{bmatrix} (2x_1+x_3) + (2y_1+y_3) \\ (-4x_2) + (-4y_2) \end{bmatrix} \text{ — (4)}$$

From (3) and (4), ~~[illegible]~~

$\Rightarrow T(X+Y) = T(X) + T(Y)$ — (1) Proved

Also, $T(cx) = T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right)$

$$= \begin{bmatrix} 2(cx_1) + cx_3 \\ -4(cx_2) \end{bmatrix} = \begin{bmatrix} c(2x_1 + x_3) \\ c(-4x_2) \end{bmatrix}$$

$$= c\begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} = c\,T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$\Rightarrow T(cx) = c\,T(x)$ —(2) Proved.

(1) and (2) are true. $\Rightarrow$ T is linear.

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