Differential Equations

Total Differential Equations

Form $\boxed{P\,dx + Q\,dy + R\,dz = 0}$

P, Q, R are constants or functions of x, y and z.

Condition for a total differential equation to be integrable :

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) =$$

i.e. $$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

Method for solving

1. If P, Q, R are homogeneous then put $x = zu$, $y = zv$ [$\Rightarrow dx = z\,du + u\,dz$, $dy = z\,dv + v\,dz$] in the given equation and solve by separating them.

2nd method Put any variable out of x, y and z constant. Let z = constant i.e. $dz = 0$.

Putting this in the given equation and solving it, we get the integral in function of z. Differentiating it and equating it with the given equation, eqn can be solved.

3rd method By Inspection.

Solve $(y^2+yz)dx + (zx+z^2)dy + (y^2-xy)dz = 0$

Solution

The given equation

$$(y^2+yz)dx + (zx+z^2)dy + (y^2-xy)dz = 0 \quad -(1)$$

It is of the form $P\,dx + Q\,dy + R\,dz = 0$

Here, $P = y^2+yz$, $Q = zx+z^2$, $R = y^2-xy$.

Now, we shall verify the condition for integrability; i.e. we prove that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

$$LHS = (y^2+yz)\left[x+2z-2y+x\right] + (zx+z^2)\left[-y-y\right]$$
$$+ (y^2-xy)(2y+z-z)$$

$$= y(y+z)(2x-2y+2z) - 2yz(x+z)$$
$$+ 2y^2(y-x)$$

$$= 2y\left[(y+z)(x-y+z) - z(x+z) + y(y-x)\right]$$

$$= 2y\left[y(\cancel{x}-\cancel{y}+\cancel{z}) + z(\cancel{x}-\cancel{y}+\cancel{z}) - \cancel{zx} - \cancel{z^2} + \cancel{y^2} - \cancel{xy}\right]$$

$$= 0 = RHS.$$

So, the equation is integrable.

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Here, P, Q and R are homogeneous.

Let $x = zu,\ y = zv$ ——— (2)

$\Rightarrow dx = z\,du + u\,dz,\ dy = z\,dv + v\,dz$ ——— (3)

using (2) and (3) in (1), we get

$$(z^2v^2 + z^2v)(z\,du + u\,dz) + (z^2u + z^2)(z\,dv + v\,dz)$$
$$+ (z^2v^2 - z^2uv)\,dz = 0$$

$$\Rightarrow z^2 v(v+1)\,z\,du + z^2 vu(v+1)\,dz + z^2(u+1).z\,dv$$
$$+ z^2(u+1)v\,dz + z^2 v(v-u)\,dz = 0$$

Dividing both sides by z^2, we get

$$\Rightarrow v(v+1)z\,du + z(u+1)\,dv + [vu(v+1) + v(u+1) + v(v-u)]\,dz = 0$$

$$\Rightarrow v(v+1)z\,du + z(u+1)\,dv + v\,dz[u(v+1) + \cancel{u}+1 + v - \cancel{u}] = 0$$

$$\Rightarrow v(v+1)z\,du + z(u+1)\,dv + v(v+1)(u+1)\,dz = 0$$

Dividing both sides by $z(u+1)v(v+1)$, we get

$$\Rightarrow \frac{du}{u+1} + \frac{dv}{v(v+1)} + \frac{dz}{z} = 0$$

$$\Rightarrow \frac{du}{u+1} + \left(\frac{1}{v} - \frac{1}{v+1}\right)dv + \frac{dz}{z} = 0.$$ Integrating, we get

$$\Rightarrow \log(u+1) + \log v - \log(v+1) + \log z = \log c$$

$$\Rightarrow vz(u+1) = c(v+1)$$ [But from (2) $u = \frac{x}{z},\ v = \frac{y}{z}$]

$$\Rightarrow y\left(\frac{x}{z} + 1\right) = c\left(\frac{y}{z} + 1\right) \Rightarrow \boxed{y(x+z) = c(y+z)}$$

Soln.

Q. Solve

$(2x^2+2xy+2xz^2+1)dx + dy + 2z\,dz = 0. \quad \text{---(1)}$

Soln First we check the condition for integrability:

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

$$LHS = (2x^2+2xy+2xz^2+1)(0-0)+(0-4xz)$$
$$+2z(2x-0) = 0 = RHS.$$

So, the given equation is integrable.

$\boxed{\text{Method}}$ ONE VARIABLE CONSTANT

Put x = constant $\Rightarrow dx = 0$

So, the given eqn becomes

$dy + 2z\,dz = 0$. Integrating, we get

$$y + z^2 = f(x) \quad \text{———} (2)$$

Now, differentiating (2), we get

$$dy + 2z\,dz - f'(x)\,dx = 0 \quad \text{———} (3)$$

comparing it with the given eqn, we get

$$\therefore f'(x) = -(2x^2+2xy+2xz^2+1)$$

$$\Rightarrow -\frac{df}{dx} = 2x^2+2x(y+z^2)+1 = 2x^2+2xf+1$$

$$\Rightarrow \frac{df}{dx}+2xf = -(2x^2+1) \quad \text{———} (4)$$

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It is a linear equation in f.

$$\therefore \; IF = e^{\int 2x\,dx} = e^{x^2}.$$

$\therefore$ Solution of the linear eqn (4) is given by

$$f \times IF = \int (-)(2x^2+1)\, IF \, dx$$

$$\Rightarrow f \cdot e^{x^2} = -\int (2x^2+1) e^{x^2} dx$$

$$= -\left[2\int x^2 e^{x^2} dx + \int e^{x^2} dx\right]$$

$$= -\left[\int x \cdot (2x e^{x^2}) dx + \int e^{x^2} dx\right] \quad (5)$$

$$\boxed{\int 2x e^{x^2} dx = \int e^{m} dm, \text{ where } m = x^2 \; = e^{m} = e^{x^2}}$$

$$\Rightarrow f \cdot e^{x^2} = -\left[x\int (2x e^{x^2}) dx - \int 1 \cdot \left\{\int 2x e^{x^2} dx\right\} dx + \int e^{x^2} dx\right]$$

$$= -\left[x \cdot e^{x^2} - \cancel{\int e^{x^2} dx} + \cancel{\int e^{x^2} dx}\right] + K$$

$$\Rightarrow (y+z^2) e^{x^2} = -x e^{x^2} + K$$

$$\Rightarrow (x+y+z^2) e^{x^2} = K$$

Q. Solve

$$(2x^2+2xy+2xz^2+1)dx + dy + 2z\,dz = 0$$

by method of inspection.

Soln The given equation

$$(2x^2+2xy+2xz^2+1)dx + dy + 2z\,dz = 0$$

$$\Rightarrow (2x^2+2xy+2xz^2)dx + dx + dy + 2z\,dz = 0$$

$$\Rightarrow 2x(x+y+z^2)dx + (dx+dy+2z\,dz) = 0$$

$$\Rightarrow 2x\,dx + \frac{dx+dy+2z\,dz}{x+y+z^2} = 0$$

Integrating, we get

$$\Rightarrow x^2 + \log(x+y+z^2) = \log K$$

$$\Rightarrow \cancel{\log} \boxed{(x+y+z^2)e^{x^2} = K}$$

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