TOPOLOGY
PAPER - IX
SEM. II
CBCS

TOPOLOGICAL SPACE

Alok Ku

By

ALOK KUMAR,
PROFESSOR & HEAD
Deptt. of Mathematics,
Maharaja College, Ara.

Definition Let X be a non-empty set. Let T be a collection of subsets of X. Then (X, T) is called a topological space if the following axioms are satisfied:

1. $X \in T$, $\phi \in T$
2. Intersection of a finite numbers of sets of T is again a set in T,

i.e. $\bigcap_{i=1}^{K} A_i \in T$ where $A_i \in T$, $i = 1$ to K

or If $A, B \in T$ then $A \cap B \in T$.

3. Union of any number of sets in T is again in T

i.e. $\bigcup_i A_i \in T$.

T is called a topology for X.

We can denote the topological space (X, T) by X. The members of T are called T-open sets or simply, open sets.

EXAMPLES

1. Let $X = \{a, b, c\}$, and

$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$

verification

(i) $X \in T$, $\phi \in T \Rightarrow$ 1st axiom is satisfied.

(ii) $X \cap \phi = \phi \in T$, $X \cap \{a\} = \{a\} \in T$,

$\phi \cap \{a\} = \phi \in T$ $X \cap \{b\} = \{b\} \in T$,

$X \cap \{a, b\} = \{a, b\} \in T$

$\{a\} \cap \{b\} = \phi \in T$ $\{a\} \cap \{a, b\} = \{a\} \in T$

$\{b\} \cap \{a, b\} = \{b\} \in T$

$\Rightarrow$ 2nd axiom is satisfied.

(iii) $X \cup \phi = X \in T$, $X \cup \{a\} = X$, $X \cup \{a\} \cup \{b\} = X$

$\phi \cup \{a\} = \{a\} \in T$, $\phi \cup \{b\} = \{b\} \in T$, $\phi \cup \{a, b\} = \{a, b\}$

$\{a\} \cup \{b\} \cup \{a, b\} = \{a, b\} \in T$

$\Rightarrow$ 3rd axiom is satisfied.

Thus, all three axioms are satisfied.

So, T is a topology on X.

2. Let $X = \{1,2,3\}$, and

$T = \{\phi, X, \{1\}, \{2\}\}$

Verification

(i) $X \in T$, $\phi \in T$ $\Rightarrow$ 1st axiom is satisfied.

(ii) $\phi \cap X = \phi \in T$, $\phi \cap \{1\} = \phi \in T$,

$\phi \cap \{2\} = \phi \in T$, $X \cap \{1\} = \{1\} \in T$,

$X \cap \{2\} = \{2\} \in T$, $\{1\} \cap \{2\} = \phi \in T$,

$\Rightarrow$ 2nd axiom is satisfied.

(iii) $\phi \cup X = X \in T$, $\phi \cup \{1\} = \{1\} \in T$,

$\phi \cup \{2\} = \{2\} \in T$, $X \cup \{1\} = X \in T$,

$X \cup \{2\} = X \in T$, ~~$\{1\}$~~

But $\{1\} \cup \{2\} = \{1,2\} \notin T$

So 3rd axiom is not satisfied.

Hence T is not a topology on X.

DISCRETE TOPOLOGY

Let X be a non-empty set.

Let T = collection of all subsets of X, that is, power set of X.

Then (X, T) is called a discrete topological space. In general, a discrete topological space is denoted by (X, D) where D denotes discrete.

Indiscrete topology

Let X be a non-empty set.

Let $T = \{X, \phi\}$

Then X, T is called indiscrete topological space. In general, an indiscrete topo. space is denoted (X, I) where I stands for indiscrete space.

TRIVIAL TOPOLOGIES

Discrete and Indiscrete topologies are called trivial topologies. other topologies are called <u>NON-TRIVIAL TOPOLOGIES</u>.

co-finite topology

Let X be a non-empty set. ~~and~~

Let T be a family of subsets of X, consisting of ϕ, ~~X~~ and all those non-empty subsets of X whose complements are finite.,

i.e. $T = \{\phi$ and $A : A$ is non-empty and A^c is finite$\}$

Thus, (X, T) is called a cofinite topology.

co-countable topology

Let X be a non-empty set.

Let T be a family of subsets of X consisting of ϕ and all those non-empty subsets of X whose complements are countable.

Then, (X, T) is called a co-countable topology.

Weak (or coarser or smaller) topology

Let X be a non-empty set. If T_1 and T_2 are two topologies on X such that $T_1 \subseteq T_2$ then we say that

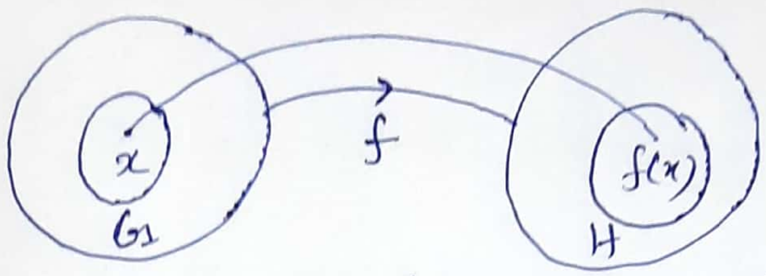
T_1 is weaker than T_2

or T_2 is stronger than T_1.

The indiscrete topology is the weakest and discrete topology is the strongest.

Continuity

Let (X, T_1) and (Y, T_2) be two topological spaces. Let $f : X \to Y$.



Then f is said to be continuous at a point $x \in X$ if for every T_2 open set H containing $f(x)$, $\exists$ a T_1 open set G containing x such that

$$f(G) \subset H.$$

f is also called $T_1 - T_2$ continuous or simply, continuous at $x \in X$.

* If f is continuous at all points of X, then f is said to be continuous at the set X.

Theorem Let (X, T_1) and (Y, T_2) be two topological spaces. Prove that a function $f: X \to Y$ is continuous iff the inverse image under f of every T_2 open set is T_1 open.

Solution

Necessary part Let $f: X \to Y$ is continuous.

Let H be a T_2 - open set.

To Prove, $f^{-1}(H)$ is a T_1 - open set.

Proof : Let $f^{-1}(H) = \phi$

Since ϕ is T_1 - open $\Rightarrow f^{-1}(H)$ is T_1 - open.
Proved.

If $f^{-1}(H) \neq \phi$ then let $x \in f^{-1}(H)$ such that $f(x) \in H$.

Given that f is continuous

$\Rightarrow$ f is continuous at $x \in X$.

$\Rightarrow$ $\exists\, G \in T_1$ such that $x \in G$ and $f(G) \subset H$.

$\Rightarrow$ $f(G) \subset H \Rightarrow G \subset f^{-1}(H)$

$\Rightarrow$ $x \in G \subset f^{-1}(H)$.

$\Rightarrow$ $f^{-1}(H)$ is a nbd of each of its points.

$\Rightarrow$ $f^{-1}(H)$ is T_1 - open. [Necessary part proved]

Sufficient part

Given that $f^{-1}(H)$ is open in X for each open set $H \subset Y$.

To Prove, f is continuous.

Proof : Given that $f^{-1}(H)$ is open in X.

Let ~~Axx~~ $G = f^{-1}(H)$ such that $G \in T_1$.

i.e. $f(G) = f[f^{-1}(H)] \subset H$

$\Rightarrow f(G) \subset H$ ~~$\Rightarrow$ for any~~

$\Rightarrow$ For any $H \in T_2$, $\exists\ G \in T_1$ such that $f(G) \subset H$.

$\Rightarrow f$ is a continuous map.

Proved

Open set

Let (X, T) be a topological space. Then any set $A \in T$ is called an open subset of X or simply an openset. $X - A$ is called closed set.

Neighbourhood (nbd)

Let (X, T) be a topological space. Let $A \subset X$. Let $x \in X$. Then A is called a nbd of a point x if there exists $G \in T$ such that $G \subset A$.

Also, let $x \in A$ and $A \in T$.

So, $x \in A \subset A$

$\Rightarrow$ Every T-open set is a nbd of each of its points.

Base (or open base) (or basis)

Let (X, T) be a topological space.
Let $B \subset T$ such that $B \neq \phi$.

B is said to be a base for the topology T on X if for any given non-empty set $G \in T \Rightarrow \exists\, B_1 \subset B$ such that

$$G = \cup \{B : B \in B_1\}$$

OR

B is said to be a base for the topology T on X if

$x \in G \in T \Rightarrow \exists\, B_1 \in B$ such that $x \in B_1 \subset G$.

The elements of B are called basic open sets.

Other Important Definitions

Let X be a topological space.
An open base for X is a class of open sets with the property that every open set is a union of sets in this class.

OR

If G is an ~~arbitr~~ open set (and $G \neq \phi$) and $x \in G$ then $\exists$ a set B in the open base s.t. $x \in B \subseteq G$.
The sets in an open base are called basic open sets.

Local base

Let (X, T) be a topological space. Let $x \in X$.

A family B_x of open subsets of X is called a local base (or base for nbd system) at x for ~~the topo~~ T if

(i) any $B \in B_x \Rightarrow x \in B$

(ii) any $G \in T$ with $y \in G \Rightarrow \exists\, B \in B_x$ such that $y \in B \subset G$.

First countable space

Let (X, T) be a topo. space. If X has a countable local base at each $x \in X$ then ~~the~~ X is said to satisfy the first axiom of countability. X is called first countable space (or first axiom space)

In other words, each point of X possesses a countable local base.

Second countable space

Let (X, T) be a topological space.

If $\exists$ a countable base for T on X, then X is said to satisfy the second axiom of countability.

X is called second countable space or second axiom space.

Homeomorphism (or topological) mapping

Let (X, T_1) and (Y, T_2) be two topological spaces. Let $f : X \to Y$. Then f is said to be homeomorphism if

(i) f is one-one onto and

(ii) f is bicontinuous

(or f is open and continuous)

~~or~~ (or f and f^{-1} are continuous).

Then (X, T_1) and (Y, T_2) are said to be homeomorphic (or topologically equivalent).

It is denoted as

$$(X, T_1) \approx (Y, T_2).$$

Also, Y is said to be homeomorphic image of X or homeomorph of X.

Theorem Prove that every second countable space is always first countable space.

Proof Let (X, T) be a topological space.

Let (X,T) is second countable.

we've to prove that (X,T) is first countable.

Since (X,T) is second countable, $\exists$ a countable base B for T on X.

B is countable, $\Rightarrow B \sim N$

$\therefore B = \{B_n : n \in N\}$ ——(1)

Let x be an arbitrary element of X.

Let L_x = collection of all those members of B which contain x.

or $L_x = \{B_n \in B : x \in B_n\}$ ——(2)

Now, L_x is a subset of B and B is countable

$\Rightarrow L_x$ is countable.

$\because$ Members of B are T-open sets and $L_x \subset B$

$\Rightarrow$ members of L_x are T-open sets.

Let $G \in L_x$. ~~L_x is a collection of~~

$\Rightarrow x \in G$ [using (2)]

Also, G is T-open $\Rightarrow G \in T$

By definition of base,

$\therefore x \in G \in T \Rightarrow B_r \in \mathcal{B}$ s.t. $x \in B_r \subset G$

$\Rightarrow \exists\ B_r \in L_x$ s.t. $B_r \subset G$

$\Rightarrow$ L_x is a countable local base at x.

$\Rightarrow$ X is first countable.