Einstein's Special Theory of Relativity

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"The problems that exist in the world today cannot be solved by the level of thinking that created them."

— Albert Einstein

Please read the earlier notes on relativity carefully before reading the current note. Mainly: Newtonian relativity, Maxwell's electromagnetic equations, details of Michelson-Morley experiment.

Note: *In literature, we found that once someone asked a question to Einstein how he was influenced by Michelson-Morley experiment, he replied that he had known about the experiment by reading a book of Lorentz (published in 1885), however, he said that he had not learnt the details of the experiment. We should know that in the late 19th century, communications were weak. Internet/email/whatsApp/twitter/facebook, etc. were not invented at that time. He was working as a clerk in a patent office at Bern and no access of research journal was there. At the age of 16, he started his work on relativity and published the work in 1905, at the age of 26.*

I. OUTLINE OF THE NOTE

II. SUMMARY OF EARLIER NOTES ON RELATIVITY

- **Frame of reference**: A set of coordinates used to specify an event.

  ⇒ **Inertial Frame**: Holds law of inertia, Acceleration=0. Example-a set of axes fixed on earth, any set of axes moving with constant velocity with respect to earth, etc.

  ⇒ **Non-inertial Frame**: Acceleration=nonzero.

- **Galilean Transformation**:

  ⇒ Space-time transformation:

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}t \quad ; t' = t. \tag{1}$$

  ⇒ Velocity transformation rule:

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}. \tag{2}$$

  ⇒ Acceleration transformation rule:

$$\mathbf{a}' = \mathbf{a}. \tag{3}$$

- **Maxwell's electromagnetic equations**:

  ⇒ Not invariant under Galilean transformation.

  ⇒ The key point to note.

  ⇒ This motivated people to think beyond the earlier developed theories, i.e., the Galileo, Kepler and Newton's concepts to study natural laws and phenomena, etc.

  ⇒ It was Einstein who resolved the mystery of Galilean invariance violation of Maxwell's equations.

- **Michelson-Morley experiment**:

  ⇒ Couldn't verify fringe shift experimentally. But their experimental results became very much helpful for better understanding the Einstein's relativity theory which was published in 1905.

  ⇒ Actually, the experimental result provides *solid base for the second postulate of the special theory of relativity* which we will discuss below in this article.

III. POSTULATES

Einstein's special theory of relativity is base on the following two postulates:

1. *The Principle of relativity*: Laws of physics are same in all inertial frame of reference.
2. *The constancy of the velocity of light*: Speed of light in free space ($c = 3 \times 10^8\ m/s$) is same (constant) in all inertial reference frames.

The first postulates is the generalized form of Newtonian relativity which states that laws of mechanics are same in all inertial systems. But here it includes all the laws of nature. This also implies that *there exit no unique frame which can be treated to be at absolute rest.* Next, remember the Michelson-Morley experiment, he observed that velocity of light is same in all directions in different inertial systems, and hence the second postulate (constancy of the speed of light) is compatible with the experiment.

⇒ This theory was being able to justify all the existing experimental outcomes at the time as well as given some more insights to for new experimental outcomes and theoretical calculations/results. Till now, no experiment has disagreed with this theory.

⇒ In 2011, some particle physicist at CERN in Geneva claimed that they found neutrino travels faster that light which could have been able enough to collapse the base of the relativity theory. But soon they realized that they have made an error in their experiment.

IV. THE LORENTZ TRANSFORMATION

In this section, we obtain the space-time transformation relations between two inertial frames using the above two postulates of the special theory of relativity. To derive these transformation rules which is also called as the Lorentz transformation equations, I list here some key points:

- See Fig. (1). We consider an event denoted by point P. For $S - frame$ observer its space-time coordinates are (x, y, z, t) and for $S' - frame$ observer the same event is specified by the coordinates (x', y', z', t').

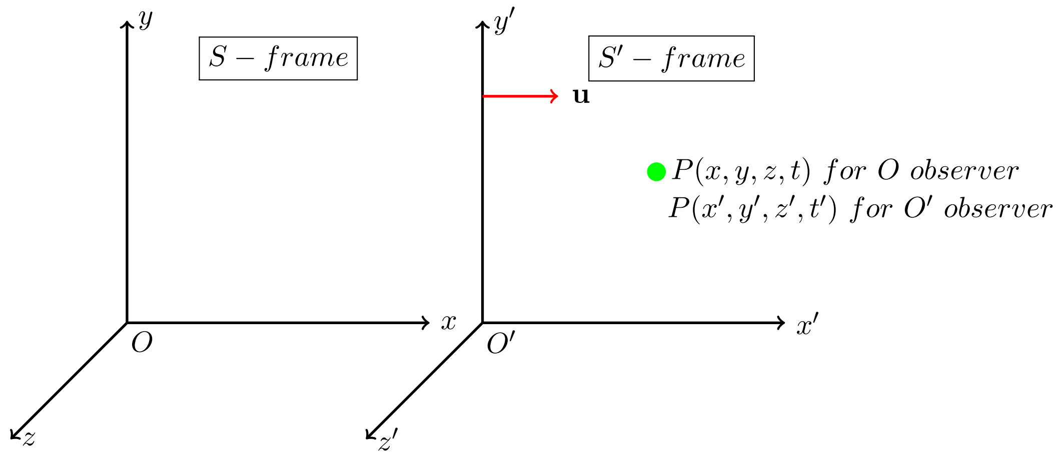

FIG. 1: S' frame moving with constant velocity $\mathbf{u}$ relative to S frame in the +ive $x-$direction, where $\mathbf{u} = (u, 0, 0)$. Both the frames have common x-x'-axis and other axes y-y' and z-z' of the two frames are parallel. Initially at $t = t' = 0$, both the frames were coinciding at the origin O.

- We need to obtain functional relations between quantities of one frame to the other frame, i.e., we need to establish relation of the form:

$$x' = x'(x, y, z, t), \quad y' = y'(x, y, z, t), \quad z' = z'(x, y, z, t) \quad \text{and} \quad t' = t'(x, y, z, t) \tag{4}$$

- Assume homogeneity of space and time ⇒ every points of space-time have identical property. We can also say that measurement of a space interval (length) and/or time interval of a particular event does not depend on the place or the time of measurement of the intervals. This implies that relations defined in Eq. (4) should be linear.

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t, \tag{5}$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t, \tag{6}$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t, \tag{7}$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t, \tag{8}$$

where the coefficients a_{ij}, for $i, j = 1, 2, 3, 4$, need to be determined to obtain the transformation equations. We have 4 equations and 16 coefficients.

- Suppose if relation are not linear, for example, take $x' = a_{11}x^2$. Now the space interval or (say) length of a rod in $S' - frame$ is given by $x'_2 - x'_1 = a_{11}(x_2^2 - x_1^2)$. If we take the ends points of the rod in the $S-frame$ to be $x_2 = 4, x_1 = 3$, then $x_2' - x_1' = 7a_{11}$. Now again if we take ends points of the rod in $S-frame$ to be $x_2 = 5, x_1 = 4$, then $x_2' - x_1' = 9a_{11}$. We see that taking nonlinear relation length is not depends on place where it is measured. So this violates the concept homogeneity of space-time. Similarly, we can give same argument for the time interval. Therefore, relations must be linear as we have considered in Eqs. (5)-(8).

- **Next, we need to determine the 16 unknown coefficients a_{ij}, for $i, j = 1, 2, 3, 4$.**

  ⇒ Since x'-axis is coinciding with the x-axis continuously. This indicates that for $y = 0, z = 0$, necessarily we have $y' = 0, z' = 0$. Therefore, we have the transformation law for y and z given as

  $$y' = a_{22}y + a_{23}z, \tag{9}$$

  $$z' = a_{32}y + a_{33}z, \tag{10}$$

  which implies that the coefficients a_{21}, a_{24}, a_{31} and a_{34} should vanish.

  ⇒ Further, $x-y$ plane $(z = 0)$ should transform to $x'-y'$ $(z' = 0)$ plane and likewise, $x - z$ plane $(y = 0)$ should transform to $x' - z'$ $(y' = 0)$ plane. Therefore, the coefficients a_{23} and a_{32} vanish and we obtain

  $$y' = a_{22}y, \tag{11}$$

  $$z' = a_{32}y. \tag{12}$$

  Next, we need to compute the coefficients a_{22} and a_{32}. Let us first consider a_{22}. We assume that a rod of unit length (i.e., $y = 1$) lying along the y-axis in the S frame. Observer in S' frame would measure the length of rod to $y' = a_{22} \times 1 = a_{22}$. Now consider that rod is brought to at rest in the primed frame $S' - frame$ along y'-axis and length will be unity for primed observer, i.e., $y' = 1$. Observer in the S- frame would measure the length of rod $y = 1/a_{22} \times 1$. For the two cases, we see that length measurements in the two frames are reciprocal in nature. But the first postulates of the relativity theory demand that measurement should be same. Therefore, for identical measurement, we should have $a_{22} = 1/a_{22} \Rightarrow a_{22} = 1$.

Thus, we obtain

$$y' = y, \tag{13}$$

$$z' = z. \tag{14}$$

⇒ We obtain Eqs. (5)-(8) as

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t, \tag{15}$$

$$y' = y, \tag{16}$$

$$z' = z, \tag{17}$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t. \tag{18}$$

⇒ Next, consider the t' equation (18), we assume that t' does not depend on y and z due to the symmetry considerations. Else a clock placed in the $y - z$ plane $(x = 0)$ about x-axis would contradict the measurement made from the S' frame. Thus a_{42} and a_{43} vanishes. We obtain

$$t' = a_{41}x + a_{44}t. \tag{19}$$

⇒ Further, consider the x' equation (15), the statement $x' = 0$ is equivalent to $x = ut$ since the point $x' = 0$ can be considered to be moving in positive x direction with speed u. Therefore $x' = a_{11}(x - ut)$ can be considered to be the correct transformation equation. Thus we write $x' = a_{11}x - ua_{11}t = a_{11}x - a_{14}t$ and obtain $a_{14} = -ua_{11}$. We obtain

$$x' = a_{11}(x - ut). \tag{20}$$

⇒ Now Eqs. (15)-(18) using (20) and (19) are obtained as

$$x' = a_{11}(x - ut) \tag{21}$$

$$y' = y, \tag{22}$$

$$z' = z, \tag{23}$$

$$t' = a_{41}x + a_{44}t. \tag{24}$$

⇒ Next, to obtain the remaining three coefficients a_{11}, a_{41}, and a_{44}, we use the second postulate of constancy of speed of light. We assume that at $t = 0$ both the frames

S and S' coincides and a spherical waves leaves the origin O. It propagates in all directions with speed c. Since it is a spherical wave so it satisfy the equation of sphere in for both frames. The radius of the sphere is given by the how much distance it traveled in some time with speed of light. For the S and S' frames the corresponding sphere equations are given by

$$x^2 + y^2 + z^2 = (ct)^2, \tag{25}$$

$$x'^2 + y'^2 + z'^2 = (ct')^2. \tag{26}$$

Using Eqs. (21)-(24) in (26) we obtain the coefficients (See AppendixA for detail calculation.) a_{11}, a_{41} and a_{44} as

$$a_{11} = \frac{1}{\sqrt{1 - u^2/c^2}}, \tag{27}$$

$$a_{44} = \frac{1}{\sqrt{1 - u^2/c^2}}, \tag{28}$$

$$a_{41} = -\frac{u/c^2}{\sqrt{1 - u^2/c^2}}. \tag{29}$$

Now using Eqs. (27)-(29) in Eqs. (21)-(24), we obtain the *Lorentz transformation equations* given by

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}, \tag{30}$$

$$y' = y, \tag{31}$$

$$z' = z, \tag{32}$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}. \tag{33}$$

By defining

$$\beta = u/c, \tag{34}$$

and

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}, \tag{35}$$

we can rewrite the Lorentz transformation equations (30)-(33) as

$$x' = \gamma(x - ut), \tag{36}$$

$$y' = y, \tag{37}$$

$$z' = z, \tag{38}$$

$$t' = \gamma(t - ux/c^2). \tag{39}$$

For $u/c \ll 1$, above equations reduces to Galilean transformation equations. Therefore, the Galilean transformation is the *special case of Einstein's relativity theory.*

The inverse Lorentz transformation: This means to say that if the corresponding quantities are observed in the S'-frame what will be the transformation rule. Simply replacing u to $-u$, we get the required transformation laws. Basically, in this case one considers that observer in the S' frame observes that S-frame is moving to left with velocity u, while observer in S-frame sees that S' frame moving to the right.

$$x = \gamma(x' + ut), \tag{40}$$

$$y = y', \tag{41}$$

$$z = z', \tag{42}$$

$$t = \gamma(t' + ux'/c^2). \tag{43}$$

V. CONSEQUENCES OF LORENTZ TRANSFORMATION RULES

In the present section we discuss the some consequences of the Lorentz transformation laws. Recall the Newtonian relativity, we have seen that length and time interval remains invariant under Galilean transformation rules. Here we see what happens to these two measurements under Lorentz transformation laws.

A. Length

Consider a rod lying at rest in the S'-frame. Its end points are x'_1 and x'_2. Length of this rod in S'-frame is $x'_2 - x'_1$. Sine we have assumed that rod is at rest in the S' frame so

$x'_2 - x'_1$ is the rest length of rod in this frame. We define that $x'_2 - x'_1 = l_0$. Now we need to calculate the length of the rod for S-frame observer for whom rod is moving with speed u. Let the S-frame observer measures the end points of the rods as x_1 and x_2, so measured length of the rod will be $x_2 - x_1 = l$ (say). Next, using Eq. (36), we can write

$$x'_1 = \gamma(x_1 - ut_1), \quad \text{and} \quad x'_2 = \gamma(x_2 - ut_2). \tag{44}$$

Or

$$x'_2 - x'_1 = \gamma[(x_2 - x_1) - u(t_2 - t_1)]. \tag{45}$$

End points of rod x_2 and x_1 in S-frame are measured at the same instant. Thus, we write $t_2 = t_1$ and obtain from Eq. (45) as

$$x'_2 - x'_1 = \gamma(x_2 - x_1). \tag{46}$$

Or

$$l_0 = \gamma l. \tag{47}$$

Now the measured length l is obtained as

$$\boxed{l = l_0/\gamma = l_0\sqrt{1-\beta^2}}\,. \tag{48}$$

Thus, *the measured length l is contracted by a factor $\sqrt{1-\beta^2}$ from the rest length l_0 of the rod.*

Note: *Length (or space interval) of an object perpendicular to the relative motion remains same for all inertial observers.*

B. Time interval

We consider a clock moving with some speed u relative to a rest frame (of course the inertial frames). We want to know what happens its rate of measurement of an interval for an even, i.e., rate remains same, slows down or becomes fast.

We consider that a clock is at rest at a given point x' in S'-frame. Let the hand of the clock spins starts at $time t'_1$ and complete one round at time t'_2. The respective times observed in S-frame is read by t_1 and t_2. Here we assume that times t_1 and t_2 are measured by two different stationary clocks of S-frame; one clock (coincident with the moving clock) is at the starting of the interval and other clock (coincident with the moving clock) corresponding to end of the interval. Thus, for observer in S-frame t_1 and t_2 are given by

$$t_1 = \gamma(t'_1 + ux'/c^2), \quad \text{and} \quad t_2 = \gamma(t'_2 + ux'/c^2), \tag{49}$$

which obtains

$$t_2 - t_1 = \gamma(t'_2 - t'_1) = \frac{(t'_2 - t'_1)}{\sqrt{1-\beta^2}}. \tag{50}$$

We define $(t'_2 - t'_1) = \tau_0$ and $t_2 - t_1 = \tau$ and obtain

$$\boxed{\tau = \gamma\tau_0 = \frac{\tau_0}{\sqrt{1-\beta^2}}}\,. \tag{51}$$

Thus, *time interval measured by observer in stationary frame gets dilated (becomes longer) by a factor of* $1/\sqrt{1-\beta^2}$ *with respect to time interval measured by moving frame observer (in which clock is at rest for the* S' *observer).* Note that clock in S' frame is moving with speed u with respect to the S frame. The time dilation can be deduced in a simpler way without using the Lorentz transformation equations. We discuss this point in the next article.

Note: *It is convention to call a frame as proper frame in which observed body is at rest. In same sense, length and time interval measured in proper frame is called respectively as proper length and proper time interval.* For example, l_0 and τ_0 are the proper length and proper time interval discussed above.

In next study material: Time dilation result without using Lorentz transformation, Addition of velocities, Doppler effect in relativity, Aberration, Mass energy relation.

Appendix A: Evaluation of coefficients

Equations of spherical waves (Eqs (25) and (26)) in the two frame is given by

$$x^2 + y^2 + z^2 = (ct)^2, \tag{A1}$$

$$x'^2 + y'^2 + z'^2 = (ct')^2. \tag{A2}$$

Writing again Eqs. (21)-(24)

$$x' = a_{11}(x - ut) \tag{A3}$$

$$y' = y, \tag{A4}$$

$$z' = z, \tag{A5}$$

$$t' = a_{41}x + a_{44}t. \tag{A6}$$

Now substituting x', y', z', and t' from Eqs. (A3)-(A6) in Eq. (A2), we obtain

$$(a_{11}^2 - c^2 a_{41}^2)x^2 + y^2 + z^2 - 2xt(ua_{11}^2 + c^2 a_{41}a_{44}) = (c^2 a_{44}^2 - u^2 a_{11}^2)t^2. \tag{A7}$$

Next, comparing Eq. (A7) with Eq. (A1), we obtain

$$a_{11}^2 - c^2 a_{41}^2 = 1, \tag{A8}$$

$$ua_{11}^2 + c^2 a_{41}a_{44} = 0, \tag{A9}$$

$$c^2 a_{44}^2 - u^2 a_{11}^2 = c^2. \tag{A10}$$

Next, write the second term on the left-hand side of Eq. (A9) to the right side and square the equation.

$$u^2 a_{11}^4 = c^4 a_{41}^2 a_{44}^2. \tag{A11}$$

Now coefficients a_{41}^2 and a_{44}^2 are substituted in Eq. (A11) in terms of a_{11}^2. Thus, it gives rise to

$$u^2 a_{11}^4 = c^4(1 + xu^2/c^2)\frac{(x-1)}{c^2}.$$

Or

$$\beta^2 a_{11}^4 = (1 + \beta^2 a_{11}^2)(a_{11}^2 - 1) = a_{11}^2 - 1 + \beta^2 a_{11}^4 - \beta^2 a_{11}^2,$$

which gives

$$a_{11} = \frac{1}{\sqrt{1-\beta^2}}, \tag{A12}$$

where we have defined $\beta = u/c$. Next, the coefficients a_{44} and a_{41} are obtained easily. There are various methods to solve these type of equations.

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[1] R. Resnick, *Introduction to Special Relativity*, Wiley-VCH, (1968) .